Lecture Notes (Week 4), Math 525 (Spring 2012)

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Application: Borsuk-Ulam theorem. Let \( Y = \mathbb{R}^2 \setminus \{0\} \).

Lemma 0.1. Let \( \gamma : I \rightarrow Y \) be a loop with \( \gamma(s + \frac{1}{2}) = \gamma(s) \) for \( s \in [0, \frac{1}{2}] \). Then \( W(\gamma) \) is even.

Proof. In short, \( \gamma \) amounts to the same loop repeated twice. Thus, think of \( \gamma = \alpha \ast \beta \) where \( \alpha \) is the evident reparameterization of \( \gamma|_{[0, \frac{1}{2}]} \), and \( \beta \) is the evident reparameterization of \( \gamma|_{[\frac{1}{2}, 1]} \). If \( x = \gamma(0) \), then both \( \alpha \) and \( \beta \) are loops based at \( x \), and in fact \( \beta = \alpha \). Thus \( W(\gamma) = 2W(\alpha) \in 2\mathbb{Z} \). \( \square \)

Lemma 0.2. Let \( \gamma : I \rightarrow Y \) be a loop with \( \gamma(s + \frac{1}{2}) = -\gamma(s) \) for \( s \in [0, \frac{1}{2}] \). Then \( W(\gamma) \) is odd.

Proof. Let \( x = \gamma(0) \), so \(-x = \gamma(\frac{1}{2})\) We can think of \( \gamma = \alpha \ast \beta \), where \( \alpha : x \sim -x \) and \( \beta(s) = -\alpha(s) \).

Let \( \tilde{\alpha} : \hat{x} \sim \hat{x} + c \) be a lift of \( \alpha \), where \( c = W(\alpha) \in (\frac{1}{2} + \mathbb{Z}) \). Then \( \tilde{\beta} : \hat{x} + c \sim \hat{x} + 2c \) given by \( \tilde{\beta}(s) = \alpha(s) + c \) is a lift a \( \beta \), and \( \hat{\gamma} = \tilde{\alpha} \ast \tilde{\beta} : \hat{x} \sim \hat{x} + 2c \) is a lift of \( \gamma \). Thus \( W(\gamma) = 2c \in (1 + 2\mathbb{Z}) \). \( \square \)

Theorem 0.3. If \( f : S^2 \rightarrow \mathbb{R}^2 \) is continuous, there exist \( x \in S^2 \) with \( f(x) = f(-x) \).

Proof. Suppose not, consider \( g : S^2 \rightarrow Y \) by \( g(x) = f(x) - f(-x) \). Note that \( g(-x) = -g(x) \).

Let \( \delta(s) = (\cos 2\pi s, \sin 2\pi s, 0) \) be a loop in \( S^2 \); it is homotopic to a constant loop, so is null homotopic, and therefore so is \( \gamma = g \circ \delta : I \rightarrow Y \). But \( \gamma(s + \frac{1}{2}) = -\gamma(s) \), so has odd winding number by the above. \( \square \)

Application: Fundamental theorem of algebra.

Theorem 0.4. Let \( f(z) = \sum_{k=0}^{n} c_k z^k \) be a polynomial with \( c_k \in \mathbb{C}, c_n \neq 0 \), and \( n \geq 1 \). Then \( f \) has a root in \( \mathbb{C} \).

Proof. WLOG, can assume \( c_n = 1 \). Observe that \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a continuous map. I’ll use the identification \( \mathbb{C} \approx \mathbb{R}^2 \).

For \( r > 0 \), let \( D_r^2 = \{ z \in \mathbb{C} \mid |z| \leq r \} \), and \( S_r^1 = \{ z \in \mathbb{C} \mid |z| = r \} \). I claim that for sufficiently large \( r \), \( f(S_r^1) \subset \mathbb{C} - \{0\} \) and \( W(f|_{S_r^1}) = n \). Since \( n \neq 0 \), this implies by the above proposition that there exists \( z_0 \in D_r^2 \) such that \( f(z_0) = 0 \).

Let \( F_t : S_r^1 \rightarrow \mathbb{C} \) be a homotopy defined by

\[
F_t(z) = (1-t)z^n + t f(z) = z^n + t \sum_{k=0}^{n-1} c_k z^k.
\]

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Theorem 0.5. If
\[\sum_{k=0}^{n-1} \left| c_k \right| z^k \leq \sum_{k=0}^{n-1} \left| c_k \right| r^k \leq \sum_{k=0}^{n-1} \left| c_k \right| r^{n-1} < r^{n-1}(|c_{n-1}| + \cdots + |c_0|) \leq r^n = |z^n|.\]

Thus \(z^n \neq t \sum_{k=0}^{n-1} c_k z^k\) for \(z \in S^1_r\), so \(F_t : S^1_r \to \mathbb{C} - \{0\}\). Thus \(F_1 = f|S^1_r\) is homotopic in \(\mathbb{C} - \{0\}\) to \(F_0 : z \mapsto z^n\), so has winding number \(n\).

\[\square\]

**Fundamental groups of products.** If \((X, x)\) and \((Y, y)\) are spaces with basepoints, there is an isomorphism
\[\pi_1(X \times Y; (x, y)) \xrightarrow{\cong} \pi_1(X; x) \times \pi_1(Y; y).\]

This amounts to the fact that continuous maps \(h : T \to X \times Y\) are the same as pairs of continuous maps \((f, g)\). So there is a one-to-one correspondence between loops in \(X \times Y\) and pairs of loops, and also a one-to-one correspondence between homotopies of paths.

**Example.** Let \(X = S^1 \times S^1\), a torus. Then \(\pi_1(X) \approx \mathbb{Z} \times \mathbb{Z}\.\)

**Fundamental group of spheres.** There’s a higher dimensional version of the Brouwer fixed point theorem: any continuous map \(f : D^n \to D^n\) has a fixed point, for any \(n \geq 0\). Unfortunately, we cannot use the fundamental group to prove this for \(n \geq 2\).

**Theorem 0.5.** If \(n \geq 2\), \(\pi_1(S^n, x_0)\) is the trivial group.

The idea of the proof is this: If \(y \in S^n\), then \(U_y = S^n - \{y\}\) is homeomorphic to an open \(n\)-disc, and so is contractible. Thus, if \(\gamma : I \to S^n\) is not surjective, it factors through some \(U_{y_i}\), and therefore is null homotopic.

The problem is that it is perfectly possible for a continuous path \(I \to S^n\) to be surjective. This is the same issue we had when \(n = 1\), but for \(n \geq 2\) the existence of space-filling curves is more surprising. (Look up “Peano curve” if you haven’t seen space-filling curves.)

However, since \(I\) is compact, and \(\{U_{y_i}\}\) is an open cover of \(S^n\), there exists a subdivision
\[0 = a_0 < a_1 < \cdots < a_n = 1\]

such that \(\gamma([a_{i-1}, a_i]) \subseteq U_{y_i}\) for some \(y_i \in S^n\).

Let \(I_i = [a_{i-1}, a_i]\), and let \(\gamma_i = \gamma|_{I_i}\). Up to reparameterization, it is a path from \(x_{i-1} = \gamma(a_{i-1})\) to \(x_i = \gamma(a_i)\) in \(U_{y_i}\).

Pick a homeomorphism \(U_{y_i} \approx \dot{D}^n\). The path \(\gamma_i : I_i \to \dot{D}^n\) is homotopic to a great circle path \(\delta_i : I_i \to \dot{D}^n\). We can glue these together, to get a path \(\delta : I \to S^n\) such that \(\delta|_{I_i} = \delta_i\), and \(\gamma \sim \delta\) as loops.

We have that \(\delta_i(I_i)\) is contained in a great circle in \(S^n\). Thus \(\delta(I)\) is contained in a finite union of great circles, which is therefore not all of \(S^n\). (This is where we use \(n \geq 2\).) Thus \(\delta : I \to S^n\) is not surjective, and we are done.

**Covering maps and covering spaces.** Let \(p : Y \to X\) be a map. Say that an open set \(U \subseteq X\) is **evenly covered** if \(p^{-1}U = \bigsqcup V_\alpha\), where the \(V_\alpha\) are open subsets of \(Y\), and \(p|V_\alpha : V_\alpha \to U\) is a homeomorphism.

We say that \(p : Y \to X\) is a **covering map** if \(X\) admits an open cover by evenly covered sets.

**Example.** \(p : \mathbb{R} \to S^1\) by \(p(s) = (\cos 2\pi s, \sin 2\pi s)\).

**Example.** \(p : \mathbb{C} \to \mathbb{C} \setminus \{0\}\) by \(p(z) = e^{2\pi i z}\).

**Example.** For \(n \neq 0\), \(p : S^1 \to S^1\) by \(p(z) = z^n\).
Example. $p: \coprod X \to X$ for any $X$. In particular, $\emptyset \to X$.
(Some texts will require for a covering map $p: Y \to X$ that $X$ and $Y$ be path connected; we’ll use this more general notion.)

Coverings of $S^1 \lor S^1$. Let $X = S^1 \lor S^1$, the one point union of two circles, which I’ll name $a$ and $b$.
Example of 2-sheeted cover. Example of infinite sheeted cover.

Coverings of $S^2 \lor S^1$. Let $X = S^2 \lor S^1$, the one point union of two circles, which I’ll name $a$ and $b$.

Example of 2-sheeted cover. Example of infinite sheeted cover.

Coverings of $S^2 \lor S^1$. Let $X = S^2 \lor S^1$. Give the simply connected covering space of this.

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Coverings from group actions. Let $G$ be a discrete group, and let $\cdot : G \times X \to X$ be a continuous action of $G$ on the space $X$. That is, $1 \cdot x = x$, $g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$, and the map is continuous.

We write $G \backslash X$ for the quotient space of $X$, using the equivalence relation $x \sim gx$ whenever $g \in G$.

Thus there is a continuous quotient map $p: X \to G \backslash X$.

Say the action is properly discontinuous if every $x \in X$ has an open neighborhood $U$ such that $U \cap gU = \emptyset$ for all $g \in G$ such that $g \neq 1$.

Note that this implies that $gx \neq x$ for all $x \in X$ and $g \neq 1$. That is, a properly discontinuous action is in particular a free action.

Exercise. If an action of $G$ on $X$ is properly discontinuous, then $p: X \to G \backslash X$ is a covering map.

Showing that an action is properly discontinuous might involve some work, but sometimes you can show it for free.

Exercise. If a finite group $G$ acts freely on a Hausdorff space $X$, then the action is always properly discontinuous.

Example. Let $G = C_2 = \langle \sigma | \sigma^2 \rangle$ act on $S^n$ by involution: $\sigma(x) = -x$. The quotient space $C_2 \backslash S^n$ is called $\mathbb{R}P^n$ (real projective $n$-space). Question: What is $\pi_1 \mathbb{R}P^n$?

(Draw model of $\mathbb{R}P^2$, obtained by making identifications on boundary of a disk.)

Example. Let $G = \{e^{2\pi i k/n} \in \mathbb{C}^\times \mid k \in \mathbb{Z}\} \subset \mathbb{C}^\times$ act on $S^{2n-1}$ as follows. We have

$S^{2n-1} \approx \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum |z_i|^2 = 1\}$

For $\lambda \in G$, we set $\lambda(z_1, \ldots, z_n) = (\lambda z_1, \ldots, \lambda z_n)$.

Example. Let $G$ be a Lie group, e.g., a closed subgroup of $GL_n(\mathbb{R})$. If $H$ is a closed subgroup of $G$ which is discrete as a space, you can show that the left action of $H$ on $G$ is properly discontinuous, and so $G \to H \backslash G$ is a covering map.

Homotopy lifting property.

Proposition 0.6. Given a covering map $p: Y \to X$, a homotopy of maps $f_t: A \to X$, and a map $\tilde{f}_0: A \to Y$ such that $p \circ \tilde{f}_0 = f_0$, there exists a unique homotopy $\tilde{f}_t: A \to Y$ such that $p \circ \tilde{f}_t = f_t$. 
That is, given \( f \) and \( \tilde{f}_0 \) there exists a unique lift \( \tilde{f}_t \) in

\[
\begin{array}{c}
A \times \{0\} \xrightarrow{\tilde{f}_0} Y \\
\downarrow \quad \downarrow \\
A \times I \xrightarrow{\tilde{f}_t} X
\end{array}
\]

When \( A = \ast \), this is just path lifting. When \( A = I \), this gives lifting for path homotopies.

**Lemma 0.7.** The proposition holds in the special case that \( f(A \times I) \subseteq U \), where \( U \) is an evenly covered open subset of \( X \).

**Proof.** This is almost the same as what we did before. The only problem is that \( A \) might not be connected.

Since \( f(A \times I) \subseteq U \), any lift must satisfy \( \tilde{f}(A \times I) \subseteq p^{-1}U \). We must show there is a unique lift in a diagram of the form

\[
\begin{array}{c}
A \times \{0\} \xrightarrow{\tilde{f}_0} p^{-1}U \\
\downarrow \quad \downarrow \\
A \times I \xrightarrow{\tilde{f}_t} U
\end{array}
\]

Write \( p^{-1}U = \bigsqcup V_\beta \), where each \( p|V_\beta: V_\beta \to U \) is a homeomorphism. The map \( \tilde{f}_0: A \to \bigsqcup V_\beta \) may map different parts of \( A \) into different “slices” \( V_\beta \). Let \( A_\beta = \tilde{f}_0^{-1}V_\beta \). Then \( \bigsqcup A_\beta \to A \) is a homeomorphism (because each \( V_\beta \) is open in \( p^{-1}U \)), and it follows that \( \bigsqcup A_\beta \times I \to A \times I \) is a homeomorphism (check this!).

Thus, it is enough to show that for each \( \beta \), there is a unique solution to the lifting problem

\[
\begin{array}{c}
A_\beta \times \{0\} \xrightarrow{\tilde{f}_0} \bigsqcup V_\beta \\
\downarrow \quad \downarrow \\
A_\beta \times I \xrightarrow{\tilde{f}_t} U
\end{array}
\]

where \( \tilde{f}_0(A_\beta) \subseteq V_\beta \). First note that any lift must satisfy \( \tilde{f}(A_\beta \times I) \subseteq V_\beta \). To see this, think about the restriction of such a lift to \( \{a\} \times I \); since this subspace is connected, its image must land in the connected component of \( \bigsqcup V_\beta \) which contains the point \( \tilde{f}_0(a) \), and this component is contained in \( V_\beta \); that is, \( (\tilde{f}_0^{-1}V_\beta) \cap (\{a\} \times I) \) will be open and closed. Thus, we can replace the map on the right with \( p|V_\beta: V_\beta \to U \), which is a homeomorphism. \( \square \)

**Proof of the homotopy lifting theorem.** We show

1. Given \( a \in A \), there exists an open neighborhood \( N \) of \( a \) and a lift \( \tilde{f}|N \times I \) of \( f|N \times I \) extending \( \tilde{f}_0|N \).
2. Given \( a \in A \), there is at most one lift \( \tilde{f}|\{a\} \times I \) of \( f|\{a\} \times I \) extending \( \tilde{f}_0|\{a\} \).

That is, there is an open cover \( \{N_a\} \) of \( A \), such that for each element of the cover there exists a solution to the lifting problem, and all such solutions must agree on overlaps. This will imply that there is a global solution, which must be unique.
Let’s prove (1). Let \( \{ U_\alpha \} \) be a cover of \( X \) by evenly covered sets. Then \( \{ f^{-1}U_\alpha \} \) is a cover of \( A \times I \). In particular, given a point \( a \in A \), we see that for each \( t \in I \), there exists

- an open neighborhood \( N_t \) of \( a \) in \( A \), and
- an open neighborhood \( B_t \) of \( t \) in \( I \), and
- an evenly covered \( U_{\alpha_t} \) open subset of \( X \), such that

\[ N_t \times B_t \subseteq f^{-1}U_{\alpha_t}. \]

Since \( I \) is compact, there exist \( 0 = t_0 < t_1 < \cdots < t_n = 1 \), and elements \( t_1^*, \ldots, t_n^* \in I \) such that

\[ [t_{i-1}, t_i] \subseteq B_{t_i^*}. \]

Taking \( N = N_{t_1^*} \cap \cdots \cap N_{t_n^*} \), we see that each \( f(N \times [t_{i-1}, t_i]) \) is contained in an evenly covered subset \( U_{\alpha_{t_i^*}} \) of \( X \). Thus, we can show there is a solution to the lifting problem over \( N \), by successively producing lifts \( \tilde{f}|N \times [t_{i-1}, t_i] \) of \( f|N \times [t_{i-1}, t_i] \) which are compatible with the already produced \( \tilde{f}_0|N \times \{ t_{i-1} \} \).

\[ \square \]