I’ll start with a brief review of topological spaces.

**Topological spaces.** Definition of topological space. \((X, T_X)\), where \(T_X\) is a collection of subsets of \(X\) (called open sets) such that

- \(\emptyset, X \in T_X\).
- \(\{U_\alpha\} \subseteq T_X \) implies \(\bigcup U_\alpha \in T_X\).
- \(U, V \in T_X\) implies \(U \cap V \in T_X\).

Complements of open sets are closed sets. Sets can be both open and closed, or neither.

**Example.** Standard topology on \(\mathbb{R}^n\).

**Example.** Discrete topology on \(X\).

Definition of continuous map. \(f: X \to Y\) is continuous if \(V\) open in \(Y\) implies \(f^{-1}V\) open in \(X\).

**Example.** Continuous maps \(\mathbb{R}^m \to \mathbb{R}^n\).

**Example.** Continuous maps \(X \to Y\) with \(X\) discrete.

**Constructions.** Ways to build new spaces, characterized by universal properties.

**Subspace.** Given a space \(X\) and a subset \(A \subseteq X\), the subspace topology on \(A\) is that given by \(U \in T_A\) if there exists \(V \in T_X\) such that \(U = A \cap V\). Let \(j: A \to X\) be the inclusion function.

*Universal property of subspace.* If \(T\) is a space, a function \(f: T \to A\) is continuous if and only if \(j \circ f\) is continuous.

*(A map \(k: Z \to X\) which factors through a homeomorphism between \(Z\) and the subspace \(k(Z) \subseteq X\) is an immersion.)*

**Quotient space.** Let \(X\) be a space, let \(B = X/\sim\) be the set of equivalence classes for some equivalence relation on \(X\), and let \(q: X \to B\) be the projection. The quotient topology on \(B\) is that given by \(U \in T_B\) if and only if \(q^{-1}U \in T_X\).

*Universal property of quotient topology.* If \(T\) is a space, a function \(g: B \to T\) is continuous if and only if \(f \circ q\) is continuous.

**Product.** Given a family \(\{X_\alpha\}_{\alpha \in I}\), the product is the space whose underlying set is the product \(X = \prod X_\alpha\), and whose topology is the smallest one for which all sets \(p_\alpha^{-1}(U_\alpha)\) are open, where \(U_\alpha \subseteq X_\alpha\) is open, and where \(p_\alpha: X \to X_\alpha\) is the projection map.

*Universal property of product.* If \(T\) is a space, a function \(f: T \to X\) is continuous if and only if \(p_\alpha \circ f\) is continuous for all \(\alpha\).

**Coproduct.** Given a family \(\{X_\alpha\}_{\alpha \in I}\), the coproduct is the space whose underlying set is the disjoint union \(X = \coprod X_\alpha\), and such that \(U \subseteq X\) is open if and only if \(U \cap X_\alpha\) is open in \(X_\alpha\) for all \(\alpha\). The inclusion maps \(i_\alpha: X_\alpha \to X\) are continuous.
Universal property of coproduct. If \( T \) is a space, a function \( g: X \to T \) is continuous if and only if \( g \circ i_\alpha \) are continuous for all \( \alpha \).

**Collapsing to a point.** Given a space \( X \) and a subspace \( A \subseteq X \), we define a space \( X/A \) as the equivalence classes of an equivalence relation \( X/\{\ast\} \), where we identify \( a \sim \ast \) for all \( a \in A \). Topologize \( X/A \) as the quotient of \( X/\{\ast\} \), which is given the coproduct topology.

A continuous map \( f: X/A \to T \) amounts to choosing a continuous map \( g: X \to T \) and a point \( t_0 \in T \) such that \( g(a) = t_0 \) for all \( a \in A \).

Note that if \( A = \emptyset \), then \( X/A = X/\{\ast\} \), and is not a quotient of \( X \). We often write \( X_+ \) for this. One way to think of this, we always want \( X/A \) to have a basepoint.

**Pushouts.** The pushout of \( Y \leftarrow A \to X \). Note that \( X/A \) is an example of a pushout \( \ast \leftarrow A \to X \).

**Examples.** Circle as interval with ends glued together. Wedge of two circles. Möbius band. Projective plane as Möbius band glued to disc.

**Homeomorphism.** A homeomorphism is a continuous map with continuous inverse.

**Example.** \( x/\sqrt{1-x^2} \colon (-1,1) \to \mathbb{R} \), with inverse \( y/\sqrt{1+y^2} \colon \mathbb{R} \to (-1,1) \). (Not same as continuous bijection, e.g., \([0,1) \to S^1\).)

**Connected and discrete.** A space \( X \) is **connected** if the only subsets which are both open and closed are \( \emptyset \) and \( X \).

**Examples.** \( \mathbb{R} \) and \( I = [0,1] \) are connected. \( \mathbb{R} \setminus \{0\} \) is not connected.

A space \( Y \) is **discrete** if every subset is open and closed. One way to characterize connectedness is: \( X \) is connected if every continuous map \( f: X \to Y \) to a discrete space \( Y \) is constant.

**Compactness.** A space \( X \) is **compact** if every open cover admits a finite subcover. The important fact we need is the Bolzano-Weierstrass theorem: the compact subspaces of \( \mathbb{R}^n \) are precisely the closed and bounded subspaces.

**Important fact.** If \( f: X \to Y \) is a continuous map and \( X \) is compact, then \( f(X) \), with the subspace topology in \( Y \), is a compact space.

This leads to the maximum principle.

**Lebesgue number lemma.** This is usually stated in terms of metric spaces. We need: if \( X \subseteq \mathbb{R}^n \) is compact, and \( \{ U_\alpha \} \) is an open cover of \( X \), then there exists \( \delta > 0 \) such that every subset of diameter \( < \delta \) is contained in the open cover. (Proof: consider all \( B_\epsilon(x;X) \) with \( x \in X \) which are contained in some element of the cover, and use compactness of \( X \).)

**Example: maps out of intervals.** Let \( I = [0,1] \) be the unit interval. If \( Y \) is a space with open cover \( \{ U_\alpha \} \), and \( f: I \to Y \) is continuous, then there exists \( N \geq 1 \) such that \( f \) sends each subinterval of the form \([ (a-1)/N, a/N ] \), where \( a \in \{1, \ldots, N \} \), into one of the \( U_\alpha \).

**Neighborhoods of compact sets of \( \mathbb{R}^n \).**

**Lemma 1.1.** If \( K \subseteq U \subseteq \mathbb{R}^n \) where \( K \) is a compact subspace of \( \mathbb{R}^n \) and \( U \) is open in \( \mathbb{R}^n \), then there exists an \( \epsilon > 0 \) such that \( K \subseteq V_\epsilon \subseteq U \), where \( V_\epsilon = \bigcup_{x \in K} B_\epsilon(x) \).

**Proof.** Consider all \( B_{2\delta}(x) \) such that \( x \in K \) and \( B_{2\delta}(x) \subseteq U \). Since \( K \) is compact, we can choose \( x_1, \ldots, x_n \in K \) and \( \delta_1, \ldots, \delta_n \) such that \( K \subseteq \bigcup B_{\delta_i}(x_i) \) and \( B_{2\delta_i}(x_i) \subseteq U \).
Let \( \epsilon = \min(\delta_i) \). If \( y \in V_\epsilon \), then there exists \( x \in K \) and \( i \in \{1, \ldots, n\} \) such that \( |y - x| < \epsilon \) and \( |x - x_i| < \epsilon \), whence \( |y - x| < 2\epsilon \leq 2\delta_i \). Thus \( y \in B_{2\delta_i}(x_i) \subseteq U \), \( V_\epsilon \subseteq U \). \( \square \)

**Spheres and disks.** Let \( D^n = \{ x \in \mathbb{R}^n \mid ||x|| \leq 1 \} \), and let \( S^{n-1} = \{ x \in \mathbb{R}^n \mid ||x|| = 1 \} \), given with subspace topologies. Note that \( S^{n-1} \) is closed in \( D^n \).

We want to construct a homeomorphism \( D^n / S^{n-1} \approx S^n \). Since \( S^n \) is defined as a subspace, and \( D^n / S^{n-1} \) is defined as a quotient space, it makes most sense to start by building a map \( f: D^n / S^{n-1} \to S^n \).

Let \( X = D^n / S^{n-1} \), and let \( q: D^n \to X \) be the quotient map. As a set, \( X \) looks like \( (D^n - S^{n-1}) \sqcup \{ \ast \} \). There is an evident continuous map \( \tilde{f}: D^n \to S^n \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1} \), given by

\[
\tilde{f}(x) = (\cos \pi ||x||, (\sin \pi ||x||)/||x|| : x)
\]

and \( \tilde{f}(0) = (1, 0) \) (since \( (1/t) \sin \pi t \) extends to a continuous function at \( t = 0 \)). Since \( \tilde{f}(S^{n-1}) = \{(-1, 0)\} \), it factors uniquely through a continuous bijection \( f: X \to S^n \).

**Claim.** \( f \) is a homeomorphism.

We must show that \( f^{-1} \) is continuous; equivalently, that \( f \) takes open sets to open sets. It is enough to have

**Proposition 1.2.** Let \( U \subseteq X = D^n / S^{n-1} \), and let \( \tilde{U} = q^{-1}U \subseteq D^n \). Then \( U \) is open in \( X \) if and only if (i) \( \tilde{U} - S^{n-1} \) is open in \( D^n \), and (ii) either \( \tilde{U} \cap S^{n-1} = \emptyset \), or there exists \( \epsilon > 0 \) such that \( V_\epsilon \subseteq \tilde{U} \), where \( V_\epsilon = \{ x \in D^n \mid ||x|| > 1 - \epsilon \} \).

**Proof.** It is clear that if \( \tilde{U} \) satisfies (i) and (ii), then it is open, and therefore \( U \) is open. Conversely, suppose \( U \) is open, and thus \( \tilde{U} \) is open; clearly \( \tilde{U} \cap (D^n - S^{n-1}) \) is open. If \( \tilde{U} \cap S^{n-1} \neq \emptyset \), then \( S^{n-1} \subseteq \tilde{U} \). Thus, there exists suitable \( \epsilon \) since \( S^{n-1} \) is compact. \( \square \)

It is clear that the restriction \( D^n - S^n \to S^n \setminus \{(-1, 0)\} \) of \( f \) is a homeomorphism, since you can describe its inverse explicitly. By the above proposition, it then suffices to show that \( f \) takes \( q(V_\epsilon) \subset D^n / S^{n-1} \) to an open set of \( S^n \), which is clear.

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