**LECTURE NOTES (PART 2), MATH 525 (SPRING 2020)**

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**Introduction to homology.** Homology is an invariant that, like the fundamental group, is a functor, and which depends only on homotopy type. It comes with theorems which are analogous to the van Kampen theorem, in the sense that you can compute $H_* (X)$ by dividing $X$ into “easy” pieces.

Original definitions of homology were by means of “triangulations”. Decompose a space into “cells”, so that the boundary of each cell is also a union of cells. If $σ$ is a cell of dimension $n$, then think of the boundary of the cell as formal sum $∂σ = \sum \pm τ_i$, where the $τ_i$ are cells of dimension $n - 1$, and the sign records something about “orientation”. If you do this right, you should discover that $∂(∂σ) = 0$, i.e., boundary of boundary is 0.

Then form a “chain complex"

$$\cdots \to C_{n+1} \xrightarrow{∂} C_n \xrightarrow{∂} C_{n-1} \to \cdots$$

where $C_n$ is the free abelian group on the $n$-cells. Then the homology group is defined as “cycles modulo boundaries” $H_n = (\text{Ker } ∂ : C_n \to C_{n-1})/(\text{Im } ∂ : C_{n+1} \to C_n)$.

This definition is not too hard, but it has problems: (i) it’s not immediately clear that homology doesn’t depend on the choice of triangulation, and (ii) it’s not obviously a functor, i.e., how should a general continuous map $f : X \to Y$ induce homomorphisms $H_n (f) : H_n (X) \to H_n (Y)$. (Remember that $π_1$ does have this functoriality property, which is key to applying it.)

Following Hatcher, we’ll implement this definition on $Δ$-complexes, and explore some of its properties. Then we give a better construction, called singular homology.

**Δ-complexes.** Let $v_0, \ldots, v_n$ be a basis for $\mathbb{R}^{n+1}$ The **standard $n$-simplex** is the subspace

$$Δ^n = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1 \text{ and } t_i \geq 0 \}.$$ That is, $Δ^n$ is the convex hull of the basis vectors $v_0, \ldots, v_n$.

Notation: if $x_0, \ldots, x_n \in \mathbb{R}^m$, then I’ll write $[x_0, \ldots, x_n]$ for the affine map $Δ^n \to \mathbb{R}^m$ sending $(t_i) \mapsto \sum t_i x_i$.

For $i = 0, \ldots, n$, let $d_i : Δ^{n-1} \to Δ^n$ be the map $[v_0, \ldots, \hat v_i, \ldots, v_n]$; the “$\hat v_i$” means to omit the term with $v_i$. These are the inclusions of the codimension 1 faces, which I’ll just call **faces**.

The **boundary** of the $n$-simplex is $∂Δ^n = \bigcup d_i(Δ^{n-1})$; these are the points $(t_0, \ldots, t_n)$ with at least one $t_i = 0$. The **interior** $\text{Int} Δ^n = Δ^n - ∂Δ^n$ is an open subset. Note that we can give a homeomorphism $Δ^n \approx D^n$, so that $∂Δ^n \approx S^{n-1}$.

A **Δ-complex structure** on $X$ is a collection $S$ of maps $σ_α : Δ^n \to X$ ($n$ depends on $α$) such that

- each $σ_α | \text{Int } Δ^n$ is injective,
- as a set $X$ is the disjoint union of the $σ_α(\text{Int } Δ^n)$,
- each restriction $σ_α \circ d^i : Δ^{n-1} \to X$ is one of the maps $σ_β : Δ^{n-1} \to X$ in the collection $S$, and
- a set $U \subseteq X$ is open if and only if $σ_α^{-1}(U)$ is open in $Δ^n$ for all $σ_α \in S$.

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Thus, a ∆-complex is a special kind of CW complex. It is one where the “attaching maps” $S^n \to X_{n-1}$ have a very explicit “combinatorial” description (i.e., are given by inclusions into cells on each of the faces).

**Examples.** ∆$^n$ itself.

Barycentric subdivision of ∆$^n$. This is another ∆-structure, where vertices are “barycenters” of simplices. That is, if $S \subseteq \{0, \ldots, n\}$ (non-empty), let $v_S$ be the point

$$v_S = \frac{1}{|S|} \sum_{s \in S} v_s \in \Delta^n.$$  

The barycentric ∆-complex structure on ∆$^n$ is the collection of maps $[v_{S_0}, v_{S_1}, \ldots, v_{S_k}]$ for each chain $S_0 \subset S_1 \subset \cdots \subset S_k$ of non-empty subsets of $\{0, \ldots, n\}$. Pictures: $n = 1$ and $n = 2$.

**Example.** Circle as quotient of 1-simplex: identify endpoints. There are only two “cells” $\sigma_0: \Delta^0 \to X$ and $\sigma_1: \Delta^1 \to X$, so that $\sigma_1 d_0 = \sigma_0 = \sigma d_1$.

**Example.** Sphere as boundary of simplex: i.e., $\partial \Delta^n$ with its evident structure inherited from $\Delta^n$.

Sphere as union of two simplices: take two copies $\Delta^n_+$ and $\Delta^n_-$, and identify along $\partial \Delta^n$.

(Note that $X = \Delta^2/\partial \Delta^2$ doesn’t work: the interior of each face of $\Delta^k \to X$ must embed in $X$. So you can “identify faces”, but you cannot “collapse” faces to smaller dimensional simplices.)

**Example.** Torus. $\mathbb{R}P^2$. Klein bottle. Note that the usual CW-complex picture of $\mathbb{R}P^2$ won’t work, and note that some 1-cells in it need to be reversed.

$$M_1 \quad N_1 \quad N_2$$

**Simplicial complex.** A simplicial complex is a ∆-complex in which

1. each map $\sigma_\alpha: \Delta^n \to X$ is injective,
2. if two maps $\sigma_\alpha, \sigma_\beta: \Delta^n \to X$ agree on vertices, then they are the same. This has the advantage of being more simple combinatorially: a simplicial complex is determined by its set of vertices $S_0$, and a collection of subsets of $S_0$ corresponding to simplices. However, it is unwieldy to calculate with.

**Simplicial homology.** Given a delta complex $X$, we define groups $\Delta_n(X)$ by

$$\Delta_n(X) = \bigoplus_{\sigma: \Delta^n \to X \in S_n} \mathbb{Z}.$$
where the sum ranges over the set $S_n$ all $n$-simplices in the $\Delta$-complex structure. Elements are finite formal sums of the form $\sum c_\sigma \sigma$, with $c_\sigma \in \mathbb{Z}$. When $n < 0$ set $\Delta_n(X) = 0$. Elements of $\Delta_n(X)$ are simplicial $n$-chains.

We define homomorphisms $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by the following formula for their action on basis elements $\sigma \in S_n$:

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i(\sigma|v_0,\ldots,\hat{v}_i,\ldots,v_n) = \sum_{i=0}^{n} (-1)^i(\sigma d^i).$$

Remember that by definition of a $\Delta$-complex, $\sigma|[v_0,\ldots,\hat{v}_i,\ldots,v_n]$ is an element of $S_{n-1}$. In low dimensions these look like

$$\begin{align*}
\partial_1(\sigma) &= \sigma|v_1) - (\sigma|v_0) \\
\partial_2(\sigma) &= \sigma|v_1,v_2) - (\sigma[v_0, v_2]) + (\sigma|v_0, v_1), \\
\partial_3(\sigma) &= (\sigma|v_1,v_2,v_3) - (\sigma|v_0, v_2,v_3) + (\sigma|v_0, v_1,v_3) - (\sigma|v_0, v_1,v_2).
\end{align*}$$

The composite

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is 0:

$$\partial_{n-1} \partial_n(\sigma) = \sum_{0 \leq j < i \leq n} (-1)^i(-1)^j(\sigma|v_0,\ldots,\hat{v}_j,\ldots,v_n) + \sum_{0 \leq i < j \leq n} (-1)^i(-1)^{j-1}(\sigma|v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n) = 0$$

**Example.** For $n = 2$, becomes

$$\begin{align*}
\partial_1\partial_2(\sigma) &= \partial_1(\sigma|v_1,v_2) - \sigma[v_0,v_2] + \sigma[v_0,v_1] \\
&= (\sigma|v_1,v_2|v_1) - \sigma[v_1,v_2][v_0]) - (\sigma[v_0,v_2][v_1] - \sigma[v_0,v_2][v_0]) + (\sigma[v_0,v_1][v_1] - \sigma[v_0,v_1][v_0]) \\
&= \sigma[v_2] - \sigma[v_1] - \sigma[v_2] + \sigma[v_0] + \sigma[v_1] - \sigma[v_0] = 0.
\end{align*}$$

The data $(\Delta_n(X), \partial_n)_{n \in \mathbb{Z}}$ is an example of a **chain complex**. That $\text{Im} \partial_{n+1} \subseteq \ker \partial_n$ follows from $\partial_n \partial_{n+1} = 0$. Notation:

$$B_n(X) := \text{im} \partial_{n+1} \subseteq Z_n(X) := \ker \partial_n \subseteq \Delta_n(X).$$

Elements of $Z_n(X) = \ker \partial_n$ are **cycles**. Elements of $B_n(X) = \text{Im} \partial_{n+1}$ are **boundaries**. We define **simplicial homology** groups by

$$H^\Delta_n(X) \overset{\text{def}}{=} \ker \partial_n / \text{Im} \partial_{n+1} = Z_n(X)/B_n(X),$$

“cycles mod boundaries”.

**Example.** Circle as quotient of $\Delta^1$. This gives the chain complex $\mathbb{Z} \overset{0}{\to} \mathbb{Z}$.

**Example.** Circle as $X = \partial \Delta^2$. This gives a chain complex

$$\mathbb{Z}\sigma_{01} \oplus \mathbb{Z}\sigma_{12} \oplus \mathbb{Z}\sigma_{02} \xrightarrow{\partial_1} \mathbb{Z}\sigma_0 \oplus \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2,$$

where I write $\sigma_{ij} = [v_i,v_j]$. The boundary map is given by

$$\begin{align*}
\sigma_{01} &\mapsto \sigma_1 - \sigma_0, \\
\sigma_{12} &\mapsto \sigma_2 - \sigma_1, \\
\sigma_{02} &\mapsto \sigma_2 - \sigma_0.
\end{align*}$$

We have $H_1X = \ker \partial_1 = \{ n(\sigma_{01} + \sigma_{12} - \sigma_{02}) \mid n \in \mathbb{Z} \} \approx \mathbb{Z}$, while $H_0X = \text{cok} \partial_1 \approx \mathbb{Z}[\sigma_0] \approx \mathbb{Z}[\sigma_1] \approx \mathbb{Z}[\sigma_2]$. 

Note that $H_1$ is a subgroup of the 1-chains here, while $H_0$ is a quotient group of the 0-chains. Thus, the generator of $H_0$ can be represented by any of the 0-chains $a\sigma_0 + b\sigma_1 + c\sigma_2$ with $a + b + c = 1$.

**Example.** $X = 2$-simplex. We just put in one more term:
\[
Z\sigma_{012} \xrightarrow{\partial_2} Z\sigma_{01} \oplus Z\sigma_{12} \oplus Z\sigma_{02} \xrightarrow{\partial_1} Z\sigma_0 \oplus Z\sigma_1 \oplus Z\sigma_2,
\]
where $\sigma_{012} = [v_0, v_1, v_2]$, so
\[
\sigma_{012} \mapsto \sigma_{01} - \sigma_{02} + \sigma_{12}.
\]
Thus $H_1X = \ker \partial_1 / \text{im} \partial_2 \approx 0$, and $H_2X \approx \ker \partial_2 = 0$. We still have $H_0X \approx \mathbb{Z}$.

**Example.** We use the $\Delta$-complex structure on $N_1 = \mathbb{R}P^2$ I described earlier. This has simplices $S_0 = \{x, y\}$, $S_1 = \{a, b, c\}$, $S_2 = \{u, v\}$, and boundary maps
\[
\begin{align*}
\partial_1(a) &= y - x, & \partial_2(u) &= a - b + c, \\
\partial_1(b) &= y - x, & \partial_2(v) &= b - a + c, \\
\partial_1(c) &= x = 0.
\end{align*}
\]
We have cycle groups $Z_0 = \mathbb{Z}x \oplus \mathbb{Z}y =, Z_1 = \mathbb{Z}(a - b) \oplus \mathbb{Z}c$, $Z_2 = \mathbb{Z}$, and boundary group $B_0 = \langle y - x \rangle$, $B_1 = \langle a - b + c, b - a + c \rangle$, $B_2 = \mathbb{Z}$. Thus $H_0 = \mathbb{Z}[x] = \mathbb{Z}[y]$, $H_1 = (\mathbb{Z}/2)[a - b]$, $H_2 = 0$. That $[a - b]$ has order 2 is because $\partial_2(u - v) = 2(a - b)$.

**Chain homotopy.** I want to prove the following.

**Proposition.** For all $n \geq 0$, $H_0^\Delta(\Delta^n) = \mathbb{Z}$, $H_q^\Delta(\Delta^n) = 0$ if $q > 0$.

Here “$\Delta^n$” means the standard $n$-simplex with its “tautological” simplicial structure, defined by the collection maps $[v_{i_0}, \ldots, v_{i_k}] : \Delta^k \rightarrow \Delta^n$ where $i_0 < \cdots < i_k$.

You can summarize this as: for all $n \geq 0$, the homology groups of $\Delta^n$ are the same as the homology of the one-point space $\{\ast\} = \Delta^0$.

Let $C_\ast$ and $D_\ast$ be chain complexes. A **chain map** is a collection of homomorphisms $f_n : C_n \rightarrow D_n$ such that $\partial_n f_n = f_{n-1} \partial_n$. (I’ll usually just write $\partial f = f \partial$.) A **chain map** $f : C_\ast \rightarrow D_\ast$ induces a map $H_\ast(f) : H_\ast C_\ast \rightarrow H_\ast D_\ast$. Explicitly, given a homology class $[c] \in H_q C$ corresponding to a cycle $c \in C_q$ with $\partial c = 0$, we define $H_\ast(f)([c]) := [f(c)]$. Verify (1) $\partial f(c) = 0$, and (2) if $c - c' = \partial(b)$, then $f(c) - f(c') = \partial f(b)$.

We need the following observation.

**Proposition.** Let $C_\ast$ and $D_\ast$ be chain complexes, and let $h = h_n : C_n \rightarrow D_{n+1}$ be a collection of homomorphisms (not a chain map). Let $f_n = \partial_{n+1} h_n + h_{n-1} \partial_n : C_n \rightarrow D_n$. Then $f$ is a chain map, and $H_\ast(f) = 0$.

**Proof.** Verify $\partial_n f_n = f_{n-1} \partial_n$, so $f$ is a chain map.

For any $c \in C_q$ with $\partial c = 0$, we have $f(c) = \partial h(c) + h \partial(c) = \partial(h(c))$, i.e., $h(c)$ is a boundary.

A **chain homotopy** between chain maps $f, g : C_\ast \rightarrow D_\ast$ is a collection of maps $h_n : C_n \rightarrow D_{n+1}$ such that $\partial h + h \partial = f - g$.

**Corollary.** If $f, g : C_\ast \rightarrow D_\ast$ are chain homotopic, then $H_\ast(f) = H_\ast(g)$.

**Proof.** The previous proposition applied to the maps $h = h_n : C_n \rightarrow D_{n+1}$ such that $f - g = \partial h + h \partial$ gives $H_\ast(f - g) = 0$, i.e., that $H_\ast(f) - H_\ast(g) = 0$. □
Now I apply this to compute the simplicial homology of $\Delta^n$. Let $h: \Delta_q(\Delta^n) \to \Delta_{q+1}(\Delta^n)$ be the homomorphism defined by

$$h([v_{i_0}, \ldots, v_{i_q}]) = \begin{cases} [v_0, v_{i_0}, \ldots, v_{i_q}] & \text{if } i_0 \neq 0, \\ 0 & \text{if } i_0 = 0. \end{cases}$$

Check that $\partial h + h \partial = \text{id}$ in all dimensions $q > 0$. On $\sigma = [v_{i_0}, \ldots, v_{i_q}]$ with $i_0 \neq 0$ and $q > 0$, we have

$$\partial h \sigma + h \partial \sigma = \left( [\sigma] - \sum_{j=0,\ldots,q} (-1)^j [v_{i_0}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_q}] \right) + \sum_{j=0,\ldots,q} (-1)^j [v_{i_0}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_q}] = \sigma,$$

while if $i_0 = 0$ we have

$$\partial h \sigma + h \partial \sigma = 0 + (\sigma + \sum_j 0) = \sigma.$$

In degree 0 we have $$(\partial h + h \partial)[v_i] = [v_i] - [v_0].$$

Thus, $h$ is a chain homotopy between $\text{id}, f: \Delta_{\bullet}(\Delta^n) \to \Delta_{\bullet}(\Delta^n)$, where $f$ is given in degree 0 by $f([v_i]) = [v_0]$, and in degree $> 0$ $f(\sigma) = 0$.

This proves that $H_q(\text{id}) = H_q(0) = 0$ on $H_q(\Delta^n)$ for $q > 0$.

(I recommend you verify this on small dimensional examples, e.g., $q = 2$. This gives a better idea of what is happening.)

**Example.** $\partial \Delta^n$, using the calculation of $H_\bullet(\Delta^n)$. For $n \geq 1$, we get $H_0(\partial \Delta^n) \approx \mathbb{Z}$, $H_{n-1}(\partial \Delta^n) \approx \mathbb{Z}$, and all other $H_q \approx 0$.

**Example.** Torus. We have

$$u \mapsto a + b - c, \quad a \mapsto 0,$$

$$v \mapsto a + b - c, \quad b \mapsto 0,$$

$$c \mapsto 0.$$

Then $Z_1 = \mathbb{Z}\{a, b, c\}$, $B_1 = \mathbb{Z}\{a + b - c\}$. We get an isomorphism

$$H_1 = Z_1/B_1 \to \mathbb{Z} \oplus \mathbb{Z}, \quad a \mapsto (1, 0), b \mapsto (0, 1), c \mapsto (1, 1).$$

Also $Z_2 = \mathbb{Z}\{a + b - c\}$, so $H_2 = Z_2 \approx \mathbb{Z}$.

**Example.** Projective plane. We have

$$u \mapsto c + a - b, \quad a \mapsto y - x, \quad v \mapsto c - a + b, \quad b \mapsto y - x,$$

$$c \mapsto 0.$$

Then $Z_1 = \mathbb{Z}\{c, a - b\}$, so $Z_1 \approx \mathbb{Z}^2$ and $B_1 = \mathbb{Z}\{c + a - b, c - a + b\}$. We get an isomorphism

$$H_1 = Z_1/B_1 \to \mathbb{Z}/2, \quad c \mapsto 1, a - b \mapsto 1.$$

Also $H_2 = Z_2 = 0$.

**Klein bottle.** We have

$$u \mapsto b + c - a, \quad a \mapsto 0,$$

$$v \mapsto a + b - c, \quad b \mapsto 0,$$

$$c \mapsto 0.$$

What is $H_1$? $H_2$?
Relative simplicial homology. Suppose $(X, A)$ is a pair consisting of a $\Delta$-complex and a subcomplex. The chain complex $\Delta_*(A)$ is naturally a subcomplex of $\Delta_*(X)$. Let $\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A)$, and $H^n_*(X, A) = H_n\Delta_*(X, A)$. Note that $\Delta_n(X, A)$ is the free abelian group on the $n$-simplices of $X$ which are not in $A$.

We obtain an exact sequence

$$0 \to \Delta_*(A) \to \Delta_*(X) \to \Delta_*(X, A) \to 0$$

of chain complexes. In each degree, there is a decomposition $\Delta_n(X) \approx \Delta_n(A) \oplus \Delta_n(X, A)$. However, we do not have a similar decomposition of homology. Instead, something more subtle happens.

Example. Consider $(X, A) = (\Delta^n, \partial \Delta^n)$, and the associated exact sequence

$$0 \to \Delta_*(\partial \Delta^n) \to \Delta_*(\Delta^n) \to \Delta_*(\Delta^n, \partial \Delta^n) \to 0.$$  

The last term is easy: $\Delta_*(\Delta^n, \partial \Delta^n)$ is 0 except for $\Delta_n = \mathbb{Z}$, and so $H_n = \mathbb{Z}$. This $\mathbb{Z}$ is the “same” as the group $H_{n-1}(\partial \Delta^n)$.

Given a short exact sequence

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

of chain complexes, the boundary homomorphism $\partial: H_n C_\bullet \to H_{n-1} A_\bullet$ is the unique homomorphism such that $\partial([z]) = [x]$ whenever there exists $y \in B_n$ such that $z = g(y)$ and $f(x) = \partial(y)$.

(Though the symbol is the same, this “boundary homomorphism” is not the same thing as a boundary operator in a chain complex.)  

Then the sequence

$$H_{n+1} C_\bullet \xrightarrow{\partial} H_n A_\bullet \xrightarrow{H_n(f)} H_n B_\bullet \xrightarrow{H_n(g)} H_n C_\bullet \xrightarrow{\partial} H_{n-1} A_\bullet$$

is exact.

Example. $(X, A)$ where $X$ is Klein bottle or $\mathbb{R}P^2$ or torus and $A$ is the “boundary of the square”.

Example. $(X, A)$, where $X$ is two $n$-simplices glued along boundary, and $A$ is one $n$-simplex.

These examples demonstrate a phenomenon called “excision”, which roughly speaking says that $H_n^\Delta(X, A)$ doesn’t depend on anything happening “inside of $A$”.

Singular homology. Singular homology is defined for arbitrary spaces, and is functorial for arbitrary continuous maps. The idea is to use every possible continuous map $\Delta^n \to X$, instead of just ones coming from some triangulation of $X$.

A singular $n$-simplex in a space $X$ is a continuous map $\sigma: \Delta^n \to X$. The singular $n$-chains are

$$C_n(X) \overset{\text{def}}{=} \bigoplus_{\sigma: \Delta^n \to X} \mathbb{Z}.$$  

The boundary map is given by $\partial \sigma = \sum_{i=0}^n (-1)^i \sigma[v_0, \ldots, \hat{v_i}, \ldots, v_n]$. The singular homology groups are $H_n X = H_n C_\bullet(X)$.

Note that if $X$ has a $\Delta$-structure, then $\Delta_*(X) \subseteq C_\bullet(X)$. We will soon show that this inclusion induces an isomorphism $H_n^\Delta(X) \approx H_n(X)$.

Singular homology is a functor. A map $f: X \to Y$ of spaces induces a chain map $f_\#: C_\bullet(X) \to C_\bullet(Y)$ by $f_\#(\sigma) = f \sigma$, and thus a map $f_* = H_* (f_\#): H_* (X) \to H_* (Y)$. Furthermore, $(gf)_* = g_* f_*$, and $id_* = id$.

We also have relative homology. Given a pair $(X, A)$ consisting of a space and a subspace,
let \( C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A) \), and \( H_\bullet(X, A) = H_\bullet C_\bullet(X, A) \). This is functorial with respect to maps \( f: (X, A) \to (Y, B) \) of pairs, which are maps \( f: X \to Y \) such that \( f(A) \subseteq B \). It is convenient to note that \( C_\bullet(\varnothing) = 0 \), so \( C_\bullet(X) = C_\bullet(X, \varnothing) \), and so \( H_\bullet(X) = H_\bullet(X, \varnothing) \). I’ll silently identify \( X \) with the pair \((X, \varnothing)\) when necessary.

Since the sequence

\[
0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \xrightarrow{j_\#} C_\bullet(X, A) \to 0
\]

is exact, there is a relative boundary operator \( \partial: H_\bullet(X, A) \to H_{\bullet-1}(A) \) for each pair.

**Exercise.** Show that boundary operator is “natural”, in the sense that if we have a “map” between two short exact sequences of chain complexes, i.e., a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & A_\bullet & \to & B_\bullet & \to & C_\bullet & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A'_\bullet & \to & B'_\bullet & \to & C'_\bullet & \to & 0
\end{array}
\]

in which the rows are exact, then the diagram

\[
\begin{array}{ccc}
H_\bullet(C_\bullet) & \xrightarrow{\partial} & H_{\bullet-1}(A_\bullet) \\
\downarrow & & \downarrow \\
H_\bullet(C'_\bullet) & \xrightarrow{\partial} & H_{\bullet-1}(A'_\bullet)
\end{array}
\]

commutes. In particular, if \( f: (X, A) \to (Y, B) \) is a map of pairs, the diagram

\[
\begin{array}{ccc}
H_\bullet(X, A) & \xrightarrow{\partial} & H_{\bullet-1}(A) \\
\downarrow f_* & & \downarrow f_* \\
H_\bullet(Y, B) & \xrightarrow{\partial} & H_{\bullet-1}(B)
\end{array}
\]

commutes.

We will show that singular homology (which is the collection \((H_q(\cdot, \cdot), \partial)\) of functors on pairs and natural boundary operators) satisfies a list of properties, which are known as the **Eilenberg-Steenrod axioms**. It turns out that two “homology theories” which satisfies these axioms must agree on a large class of spaces, which includes all spaces which are homotopy equivalent to CW-complexes (but not all spaces). Also, any such homology theory will agree with \( H_\Delta^\ast \) for any \( \Delta \)-complex, as we will see. The axioms are:

- Dimension.
- Sum.
- Exactness.
- Homotopy.
- Excision.

**Proposition** (Dimension Axiom). If \( X \) is a one point space, then \( H_0(X) \approx \mathbb{Z} \), and \( H_q(X) = 0 \) for \( q \neq 0 \).

**Proof for singular homology.** Compute the complex:

\[
\cdots \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to 0 \to \cdots
\]
Proposition (Sum Axiom). Let $X = \coprod X_\alpha$, and let $i_\alpha: X_\alpha \to X$ denote the tautological inclusion. Then
\[ \bigoplus H_s(X_\alpha) \xrightarrow{((i_\alpha)_*)} H_s X \]
is an isomorphism.

Proof for singular homology. Because $\Delta^n$ is path connected, and so connected, each $\sigma: \Delta^n$ has image in exactly one $X_\alpha$. Using this, we find that the inclusion maps induce an isomorphism of chain complexes $\bigoplus H_\bullet(X_\alpha) \xrightarrow{\sim} H_\bullet X$, which then induces the desired isomorphism in homology. \[\square\]

Actually for singular homology we can do a little better.

Proposition. Let $X = \bigcup X_\alpha$ be written as a disjoint union of its path components (but necessarily topologized as the coproduct), with inclusion maps $i_\alpha: X_\alpha \to X$. Then
\[ \bigoplus H_\bullet(X_\alpha) \xrightarrow{((i_\alpha)_*)} H_\bullet X \]
is an isomorphism.

Proof. Same as above, since each $\Delta^n$ is path connected. \[\square\]

Here’s a related fact, which also does not follow in general from the axioms.

Proposition. The group $H_0(X)$ is the free abelian group on the set of path components of $X$.

Proof. Write $X = \bigcup X_\alpha$ as a disjoint union of its path components $X_\alpha$. As I proved above, there is an isomorphism $\bigoplus H_0(X_\alpha) \xrightarrow{\sim} H_0(X)$. So it will suffice to show that if $X$ is itself path connected, then $H_0(X) \cong \mathbb{Z}$.

So assume $X$ is path connected, and consider the sequence of homomorphisms $C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\pi} \mathbb{Z} \to 0$.

Note that $C_0(X) \cong \bigoplus_{x \in X} \mathbb{Z}$ is the free abelian group on the set of points of $X$. Define $\pi$ by $\pi\left( \sum \alpha \cdot c_\alpha \right) := \sum x_c$. We know $C_1(X) \cong \bigoplus_{\sigma: \Delta^1 \to X}$ is the free abelian group on paths in $X$, and that $\partial(\sigma) = \sigma([v_1]) - \sigma([v_0])$, and so we can verify that $\pi \partial(\sigma) = 1 - 1 = 0$, i.e., $\pi \circ \partial = 0$.

To prove that $H_0(X) = C_0(X)/\partial C_1(X)$ is isomorphic to $\mathbb{Z}$, it suffices to show that the above sequence is exact. So I need to show that if $a = \sum c_x \cdot x \in C_0$ is such that $\pi(a) = 0$, then $a = \partial(b)$ for some $b \in C_1(X)$.

Since every non-zero integer is either $1 + \cdots + 1$ or $(-1) + \cdots + (-1)$, we can write $a = \sum_{i=1}^m x_i - \sum_{i=1}^n y_i$, $x_i, y_i \in X$. We have $\pi(a) = m - n$, so $\pi(a) = 0$ implies $m = n$. Since $X$ is path connected, there exist for each $i = 1, \ldots, m$ a $\sigma_i: \Delta^1 \to X$ such that $\sigma_i([v_0]) = y_i$, $\sigma_i([v_1]) = x_i$. Set $b = \sum_{i=1}^n \sigma_i$. Then $\partial(b) = a$, as desired. \[\square\]
Long exact sequence.

**Proposition** (Exactness Axiom). For any pair \((X, A)\), the sequence

\[
H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)
\]

is exact.

This is proved exactly as for simplicial homology.

**Homotopy axiom.**

**Proposition** (Homotopy Axiom). If \(f, g: X \to Y\) are homotopic maps, then \(f_* = g_*: H_*(X) \to H_*(Y)\).

**Corollary.** If \(f: X \to Y\) is a homotopy equivalence, then \(f_*: H_*(X) \to H_*(Y)\) is an isomorphism.

In particular, the map \(H_*(D^n) \to H_*(pt)\) is an iso.

**Example.** Consider the pair \((D^n, S^{n-1})\). Since \(D^n\) is contractible, the long exact sequence becomes

\[
0 = H_qD^n \xrightarrow{\partial} H_q(D^n, S^{n-1}) \xrightarrow{\partial} H_{q-1}S^{n-1} \to H_{q-1}D^n = 0
\]

for \(q \geq 2\), which implies \(H_q(D^n, S^{n-1}) \approx H_{q-1}S^{n-1}\).

**Remark.** The Homotopy Axiom implies that \(\pi: X \times I \to X\) induces an isomorphism on homology, since it is a homotopy equivalence. In fact, the Homotopy Axiom is equivalent to the fact that all such \(\pi\) induce isos on homology. Consider the inclusions \(i_0, i_1: X \to X \times I\). Since \(\pi i_0 = \text{id}_X = \pi i_1\), if \(\pi_*\) is iso then \((i_0)_* = (i_1)_*\). If \(F_i: X \to Y\) is a homotopy, then \(F_0 = F i_0\) and \(F_1 = F i_1\), whence \((F_0)_* = F_* (i_0)_* = F_* (i_1)_* = (F_1)_*\).

**Proposition.** Let \(f: (X, A) \to (Y, B)\) be a map of pairs such that \(f: X \to Y\) is a homotopy equivalence and \(f|A: A \to B\) is a homotopy equivalence. Then \(f_*: H_*(X, A) \to H_*(Y, B)\) is an isomorphism.

**Proof.** Immediate using the 5-lemma. \(\square\)

**The 5-lemma.** Consider the commutative diagram of abelian groups

\[
\begin{array}{cccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow^\alpha & & \downarrow^\beta & & \downarrow^\gamma & & \downarrow^\delta & & \downarrow^\epsilon \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}
\]

in which both rows are exact. Check that

(i) \(\beta\) and \(\delta\) injective and \(\alpha\) surjective imply \(\gamma\) injective.

(ii) \(\beta\) and \(\delta\) surjective and \(\epsilon\) injective imply \(\gamma\) surjective.

Therefore

(iii) \(\alpha, \beta, \delta, \epsilon\) iso imply \(\gamma\) iso.

We usually use this in the consequence of a long exact sequence. If \(f: (X, A) \to (Y, B)\) is a map of pairs, and two of the three maps \(H_*A \to H_*B, H_*X \to H_*Y, H_* (X, A) \to H_* (Y, B)\) are iso, so is the third.

**Exercise.** Using the axioms (not the definition of singular homology), show that if \((X, A) = \coprod (X_\alpha, A_\alpha)\), then \(H_*(X, A) \approx \bigoplus H_*(X_\alpha, A_\alpha)\).
Proof of the homotopy axiom. Fix a homotopy $F: X \times I \to Y$ between $F_0 = f$ and $F_1 = g$. Will produce a chain homotopy $h: C_k(X) \to C_{k+1}Y$ between $g\#$ and $f\#$, by giving as explicit formula. So that the proof doesn’t seem to obscure, I’ll try to explain where it comes from. The basic idea is that the formula only really involves simplices in a $\Delta$-complex structure on a “prism” $\Delta^n \times I = \Delta^n \times \Delta^1$.

Two points of view on the simplex. First, let’s make an observation about the simplex. The point is that there are two different ways of thinking about an $n$-simplex:
- It is the convex hull of $n + 1$ points in $\mathbb{R}^n$ (in general position).
- It is the configuration space of subdivisions of $[0, 1]$ into $n + 1$ subintervals.

Exercise. Recall that
$$\Delta^n := \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0 \}.$$ 

Let
$$\Delta^n_S := \{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1 \}.$$ 

Show that $\Delta^n \approx \Delta^n_S$ by
$$(x_0, \ldots, x_n) \mapsto (x_0, x_0 + x_1, \ldots, x_0 + \cdots + x_{n-1}) = (t_1, \ldots, t_n),$$
$$(x_0, \ldots, x_n) = (t_1 - 0, t_2 - t_1, \ldots, 1 - t_n) \leftrightarrow (t_1, \ldots, t_n).$$

(Picture for $n = 2$: the $t$s are the subdivision points, and the $x$s are the lengths of the subintervals.)

The $\Delta$-complex structure on $\Delta^n$ transfers to one on $\Delta^n_S$. The interior of $\Delta^n_S$ corresponds to the non-degenerate subdivisions, i.e., with $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$. The images of smaller dimensional simplices correspond to sets of subdivisions for which $t_j = t_{j+1}$ for certain values of $i$. For instance, the image of $[v_{i_0}, \ldots, v_{i_q}]$ corresponds to the set of subdivisions $0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1$ with $t_j = t_{j+1}$ for all $j \notin \{i_0, \ldots, i_q\}$.

The prism. We consider a particular $\Delta$-complex structure on the prism $\Delta^n \times I$. We can think of this as a subspace of $\mathbb{R}^{n+1} \times \mathbb{R}$, with vertices $a_k = (v_k, 0)$ and $b_k = (v_k, 1)$. The maps $\sigma_k: \Delta^n \to \Delta^n \times I$ are the ones of the form $[a_0, \ldots, a_{i_p}, b_{i_{p+1}}, \ldots, b_{i_q}]$ where $i_k < i_{k+1}$ for all $k \neq p$, and $i_p \leq i_{p+1}$; we also allow for the possibility that $p = q$ or $p + 1 = 0$. (Picture for $n = 1$, maybe $n = 2$.)

That is, the $\sigma_k$ correspond exactly to totally ordered chains in the partially ordered set of vertices $\{v_0, \ldots, v_n\} \times \{0, 1\}$. The top dimensional (i.e., $n + 1$) simplices are $\sigma_k := [a_0, \ldots, a_k, b_k, \ldots, b_n]$, $k = 0, \ldots, n$.

From the point of view of “simplex as subdivisions”, we can describe $\Delta^n_S \times I$ as the space whose points pairs
$$(t, y) = (0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1, y \in I),$$

consisting of a subdivision $t$ and a point $y \in I$. The images of the top dimensional simplex $\sigma_k$ correspond to those pairs $(t, y)$ such that $y \in [t_k, t_{k+1}]$.

I won’t show that this really gives $\Delta$-complex structure on $\Delta^n \times I$, but it is not hard. It is easier to see that $\Delta^n \times I$ is a union of the interiors of the $\sigma_k$ using the subdivision picture.

The only fact we really need is that each $\sigma_\alpha \circ d^\alpha$ is another such simplex, so that there is a well-defined simplicial chain complex $\Delta_*(\Delta^n \times I)$, which we think of as a subcomplex of $C_*(\Delta^n \times I)$.

Remark. Similarly, one can give a $\Delta$-structure on $\Delta^n \times \Delta^n$. The simplices in the structure corresponds to all ways you can “shuffle” two ordered sequences $0 \leq x_1 \leq \cdots \leq x_m \leq 1$ and $0 \leq y_1 \leq \cdots \leq y_n \leq 1$ together to get an ordered sequence.
**Prism operators.** Define maps $P : \Delta_q(\Delta^n) \to \Delta_{q+1}(\Delta^n \times I)$ by

$$P([v_{i_0}, \ldots, v_{i_q}]) = \sum_{k=0}^{q} (-1)^k [a_{i_0}, \ldots, a_{i_k}, b_{i_k}, \ldots, b_{i_q}].$$

The simplicies in the sum are exactly the ones that project onto $[v_{i_0}, \ldots, v_{i_q}]$ in $\Delta^n$ and onto $[0, 1]$ in $I$. The $P$ are called **prism operators**.

There are two maps $s, t : \Delta^n \to \Delta^n \times I$, given by $s(x) = (x, 0)$ and $t(x) = (x, 1)$. In fact, $s = [a_0, \ldots, a_n]$ and $t = [b_0, \ldots, b_n]$. These maps are compatible with the $\Delta$-complex structures, in the sense that the induced maps $C_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times I)$ restrict to chain maps on subcomplexes $s_\# , t_\# : \Delta_\bullet(\Delta^n) \to \Delta_\bullet(\Delta^n \times I)$.

The face maps $d^i = [v_0, \ldots, \hat{v}_i, \ldots, v_n] : \Delta^{n-1} \to \Delta^n$ are also compatible with $\Delta$-complex structures. Thus, we have chain maps $d^i_\# : \Delta_\bullet(\Delta^{n-1}) \to \Delta_\bullet(\Delta^n)$ and $(d^i \times \text{id})_\# : \Delta_\bullet(\Delta^{n-1} \times I) \to \Delta_\bullet(\Delta^n \times I)$.

**Exercise.** $\partial P + P \partial = t_\# - s_\#$ (just plug in $[v_{i_0}, \ldots, v_{i_q}]$ into either side and compute: I recommend you look at a low dimensional example, like $n = 1$ or $n = 2$). Thus, $P$ is a chain homotopy between $s_\#$ and $t_\#$ as chain maps $\Delta_\bullet(\Delta^n) \to \Delta_\bullet(\Delta^n \times I)$.

**Exercise.** $P d^i_\# = (d^i \times \text{id})_\# P$ (again, just plug in $[v_{i_0}, \ldots, v_{i_q}]$). Thus, the prism operators are natural with respect to face inclusions.

**Proof of homotopy axiom.** Let $F : X \times I \to Y$ be a homotopy between $f$ and $g$. We will produce a chain homotopy $H : C_n(X) \to C_{n+1}(Y)$ so that $\partial H + H \partial = g_\# - f_\#$, which will prove the result. This is done by the above construction, so that for $\sigma : \Delta^n \to X$ we define,

$$H(\sigma) := F_\#(\sigma \times \text{id})_\# P([v_0, \ldots, v_q]).$$

To show that $H$ has the desired property, observe that

$$\partial H(\sigma) = F_\#(\sigma \times \text{id})_\# \partial P([v_0, \ldots, v_q]),$$

while

$$H \partial \sigma = \sum (-1)^i H(\sigma d^i)$$

$$= \sum (-1)^i F_\#(\sigma d^i \times \text{id})_\# P([v_0, \ldots, v_{q-1}])$$

$$= \sum (-1)^i F_\#(\sigma \times \text{id})_\# (d^i \times \text{id})_\# P([v_0, \ldots, v_{q-1}])$$

$$= \sum (-1)^i F_\#(\sigma \times \text{id})_\# P d^i_\# ([v_0, \ldots, v_{q-1}])$$

$$= F_\#(\sigma \times \text{id})_\# P \partial([v_0, \ldots, v_q]).$$

Thus,

$$\partial H + H \partial(\sigma) = F_\#(\sigma \times \text{id})_\# (\partial P + P \partial)([v_0, \ldots, v_q])$$

$$= F_\#(\sigma \times \text{id})_\# (t_\# - s_\#)([v_0, \ldots, v_q]) = (g_\# - f_\#)(\sigma),$$

since $F(\sigma \times \text{id})t = g\sigma$ and $F(\sigma \times \text{id})s = f\sigma$. \qed

**Excision.**

**Theorem (Excision Axiom).** Let $(X, A)$ be a pair, and suppose $Z$ is such that $\text{Cl} Z \subseteq \text{Int} A$. Then the inclusion map $i : (X - Z, A - Z) \to (X, A)$ induces an isomorphism $i_* : H_*(X - Z, A - Z) \to H_*(X, A)$.

Here is another way to state this. Let $B = X - Z$. Then this says that if $X = \text{Int} A \cup \text{Int} B$, then $(B, B \cap A) \to (X, A)$ induces an isomorphism in homology.
Let \( \mathcal{U} = \{U_\alpha\} \) be a collection of subsets of \( X \) such that \( X = \bigcup \text{Int } U_\alpha \). Let \( C_n^\mathcal{U}(X) \subseteq C_n(X) \) be the subgroup spanned by singular simplices \( \sigma : \Delta^n \to X \) whose image is contained in one of the \( U_\alpha \). This defines a subcomplex \( C_n^\mathcal{U}(X) \). Excision follows from the following.

**Proposition.** Inclusion induces an isomorphism \( H_\ast C^\mathcal{U}_\ast(X) \approx H_\ast C_\ast(X) \).

To prove Excision using this, take \( \mathcal{U} = \{A, X - Z\} \). There is a commutative diagram of chain complexes

\[
\begin{array}{cccccc}
0 & \longrightarrow & C_\ast(A) & \longrightarrow & C_\ast^\mathcal{U}(X) & \longrightarrow & C_\ast(X - Z, A - Z) & \longrightarrow & 0 \\
0 & \longrightarrow & C_\ast(A) & \longrightarrow & C_\ast(X) & \longrightarrow & C_\ast(X, A) & \longrightarrow & 0 \\
\end{array}
\]

in which the rows are exact. The result follows using the 5-lemma.

To prove the proposition, we need the technique of “barycentric subdivision”. (Draw pictures.)

We will construct for each space \( X \)

(a) a chain map \( S : C_\ast(X) \to C_\ast(X) \), so \( \partial S = S \partial \), and

(b) a chain homotopy \( T : C_\ast(X) \to C_\ast+1(X) \) such that \( \partial T + T \partial = \text{id} - S \),

such that

(i) the functions \( S \) and \( T \) are natural with respect to continuous maps \( f : X \to Y \), i.e., \( Sf_\# = f_\# S \) and \( Tf_\# = f_\# T \), and

(ii) for any cover \( \mathcal{U} \), and every \( \sigma : \Delta^n \to X \), there exists an \( m \) such that \( S^m \sigma \in C^\mathcal{U}_\ast(X) \).

Here are two immediate consequences.

(iii) The functions \( S \) and \( T \) on \( C_\ast(X) \) restrict to functions from \( C^\mathcal{U}_\ast(X) \) to itself. (This follows from naturality.)

(iv) The chain map \( S^m \) is chain homotopic to the identity, by the chain homotopy \( T_m = T(1 + S + \cdots + S^{m-1}) \), so that \( \partial T_m + T_m \partial = \text{id} - S^m \).

Here’s how this proves the proposition.

First, let’s show that \( H_\ast(i) : H_\ast C^\mathcal{U}_\ast(X) \to H_\ast C_\ast(X) \) is surjective. Suppose that \( a \in C_\ast(X) \) such that \( \partial(a) = 0 \). Since \( a \) is a finite sum of simplices, there exists an \( m \) such that \( S^m a \in C^\mathcal{U}_\ast(X) \). Let \( a' = S^m a \), let \( b = T_m a \in C_{n+1}(X) \), and observe that

\[
\partial(a') = \partial S^m(a) = S^m \partial(a) = 0, \quad \partial(b) = \partial T_m(a) = a - S^m(a) - T_m \partial(a) = a - a'.
\]

So \( [a'] \in H_\ast C^\mathcal{U}_\ast(X) \) is such that \( i_*([a']) = [a] \).

Next, let’s show that \( H_\ast(i) \) is injective. Suppose that \( a \in C^\mathcal{U}_\ast(X) \) such that \( \partial(a) = 0 \), and suppose there exists \( b \in C_{n+1}(X) \) such that \( \partial(b) = a \). Let \( b' = S^m(b) + T_m(a) \), which is an element of \( C^\mathcal{U}_{n+1}(X) \) if \( m \) is large enough, and observe that

\[
b' = S^m(b) + T_m(a) = S^m(b) + T_m \partial(b) = b - \partial T_m(b),
\]

and thus

\[
\partial(b') = \partial(b - \partial T_m(b)) = \partial(b) = a.
\]

So if \( [a] \in H_\ast C^\mathcal{U}_\ast(X) \) is such that \( i_*([a]) = 0 \), then \( [a] = 0 \), since \( a = \partial(b') \) with \( b' \in C^\mathcal{U}_{n+1}(X) \).

**Construction of \( S \) and \( T \): barycentric subdivision.** To construct the operators \( S \) and \( T \), we need to think about “linear chains”. Suppose that \( X \) is a convex subset of a Euclidean space \( \mathbb{R}^n \). Let \( L_q(X) \subseteq C_q(X) \) be the span of all affine linear maps of the form \( [x_0, \ldots, x_q] : \Delta^q \to X \); this is a subcomplex. Linear chains are natural with respect to linear maps, i.e., affine linear transformations \( f : X \to Y \) induce \( f_\# : L_q(X) \to L_q(Y) \). (We will only actually care about \( X = \Delta^n \) and \( f = d^p \).)
A point $x \in X$ determines a homomorphism $x : L_q(X) \to L_{q+1}(X)$, by $x \cdot [x_0, \ldots, x_q] = [x, x_0, \ldots, x_q]$; the new simplex is the “cone” on the original one, with $x$ as the cone point. Observe that for a linear simplex $\sigma \in L_q(X)$ with $q \geq 1$, we easily check
\[ \partial(x \cdot \sigma) + x \cdot \partial(\sigma) = \sigma; \]
if $q = 0$, we have $\partial(x \cdot \sigma) + x \cdot \partial(\sigma) = \sigma - [x]$. You can read this as saying “the boundary of the cone on a simplex is the original simplex, together with the cone of its boundary”. (This shows that $H_*(L_*(X)) \approx H_*(pt)$, by the way: the operation $x \cdot -$ is a chain homotopy between $id_\#$ and $f_\#$ where $f(y) = x$ for all $y$.)

Given a map $\sigma : \Delta^n \to X$, let $b_\sigma = \sigma(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. This is the barycenter.

Define $S : L_q(X) \to L_q(X)$ inductively, on linear singular simplices $\sigma$ by $S([x]) = [x]$ if $q = 0$, and for $q > 0$,
\[ S\sigma = b_\sigma \cdot S(\partial \sigma). \]
(Give examples and draw pictures.) Check that
\[ \partial S = S \partial. \]

Define $T : L_q(X) \to L_{q+1}(X)$ inductively, on linear singular simplices $\sigma$ by $T([x]) = 0$ if $q = 0$, and for $q > 0$,
\[ T\sigma = b_\sigma \cdot (\sigma - T \partial \sigma). \]
Observe that
\[ \partial T = 1 - S - T \partial. \]

Note that $S$ and $T$ are natural with respect to linear maps $f : X \to Y$, i.e., $f_\# S = S f_\#$ and $f_\# T = T f_\#$. This is because the “cone” operation is also natural with respect to linear maps: $f_\#(x \cdot \sigma) = f(x) \cdot f_\#(\sigma)$, and because barycenter is natural with respect to affine linear maps, i.e., $b_{f_\# \sigma} = f(b_\sigma)$. In particular, they are natural with respect to boundary inclusions $d^i : \Delta^{n-1} \to \Delta^n$.

Finally, observe that if $\sigma : \Delta^n \to X$ is a linear singular simplex of diameter $d$, then $S\sigma = \sum n_i \sigma_i$ is such that each $\sigma_i$ has diameter $d(n/(n+1))$. Thus, if $U$ is a cover of $X$, then there exists $m$ such that $S^m \sigma$ is contained in $L_n(X) \cap C^m_\#(X)$.

Now define operators $S$ and $T$ on $C_\#(X)$ as follows. Let $v_q = [v_0, \ldots, v_q] \in L_q(\Delta^q)$ be the tautological element. Thus, a simplex $\sigma : \Delta^n \to X$ corresponds to the element $\sigma_\#(v_q) \in C_\#(X)$.

Define
\[ S\sigma \overset{\text{def}}{=} \sigma_\#(v_q), \quad T\sigma \overset{\text{def}}{=} \sigma_\#(T v_q). \]
Checking that these satisfy the desired formulas, and are natural, is straightforward. Likewise, if $\sigma : \Delta^n \to X$ is a map and $U = \{U_\alpha\}$ is a cover of $X$, then $\{\sigma^{-1} U_\alpha\}$ is a cover of $\Delta^n$.

**Homology of spheres from Eilenberg-Steenrod axioms.** Decompose $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$. There is an inclusion of pairs $f : (D^n_+, S^n-1) \to (S^n, D^n_-)$.

**Proposition.** This map induces an isomorphism $f_* : H_*(D^n_+, S^n-1) \to H_*(S^n, D^n_-)$ on all homology groups.

We might like to use excision here, since we are “removing” $\text{Int}(D^n_-)$, but $\text{Int} D^n_+ \cup \text{Int} D^n_- \neq S^n$. Instead, let $U$ be a neighborhood of $D^n_-$ in $S^n$ which deformation retracts to $D^n_-$, and consider the square
\[
\begin{array}{ccc}
(D^n_+, S^n-1) & \xrightarrow{t} & (D^n_+ \cap D^n_-) \\
\downarrow f & & \downarrow f \\
(S^n, D^n_-) & \xrightarrow{t} & (S^n, U)
\end{array}
\]
You can picture $U$ as only slightly larger than $D^n$, though in fact we could use $U = S^n$ (north pole). The horizontal maps induce isomorphisms in homology (in all degrees) using homotopy, because $S^{n-1} \subset U \cap D^n_+$ and $D^n_+ \subset U$ are homotopy equivalences. The right vertical map induces isomorphisms in homology (in all degrees) by excision, since $\text{Int } D^n_+ \cup \text{Int } U = S^n$. Thus the left vertical map induces an isomorphism in homology.

\[
\begin{array}{ccc}
H_*(D^n_+, S^{n-1}) & \xrightarrow{\text{htpy}} & H_*(D^n_+, U \cap D^n_+) \\
\downarrow f_* & & \downarrow \sim & \text{excision} \\
H_*(S^n, D^n_-) & \xrightarrow{\text{htpy}} & H_*(S^n, U)_*
\end{array}
\]

Notice what happened here: we don’t just learn that $H_*(D^n_+, S^{n-1}) \approx H_*(S^n, D^n_-)$, but that this isomorphism is induced by the map $f_*$, even though excision did not apply directly to $f$.

Recall the map of pairs, given by

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{f} & D^n_- \xrightarrow{f} (D^n_+, S^{n-1}) \\
\downarrow j & & \downarrow j & \downarrow j \\
D^n_+ & \xrightarrow{f} S^n \xrightarrow{f} (S^n, D^n_-)
\end{array}
\]

inducing isos $H_*(D^n_-, S^{n-1}) \xrightarrow{\sim} H_*(S^n, D^n_-)$. Now we compute the homology of $S^n$ by induction on $n$, using the zig-zag of maps

\[
H_q(S^n) \xrightarrow{j_*} H_q(S^n, D^n_-) \xrightarrow{f_*} H_q(D^n_+, S^{n-1}) \xrightarrow{\partial} H_{q-1}(S^{n-1}).
\]

In most cases, all these maps are isomorphisms, because of the long exact sequence of pairs and the fact that the homology of $D^n$ is basically trivial. Calculate homology of spheres using this, starting with $n = 0$.

- When $n = 0$, we know the answer: $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_q(S^0) = 0$ if $q \neq 0$.
- When $n = 1$, we use the exact sequences for $(D^1_+, S^0)$ and $(S^1, D^1_-)$ and the map $f$:

\[
\begin{array}{ccc}
H_1S^0 & \xrightarrow{i_*} & H_1D^1_+ \xrightarrow{j_*} H_1(D^1_+, S^0) \xrightarrow{\partial} H_0S^0 \\
\downarrow f_* & & \downarrow f_* & \sim \\
H_1D^1_- \xrightarrow{i_*} H_1S^1 \xrightarrow{j_*} H_1(S^1, D^1_-) \xrightarrow{\partial} H_0D^1_-
\end{array}
\]

Filling in what we know, we get

\[
\begin{array}{ccc}
0 & \xrightarrow{i_*} & 0 \xrightarrow{j_*} H_1(D^1_+, S^0) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z} \\
\downarrow f_* & & \downarrow f_* & \sim \\
0 & \xrightarrow{i_*} & H_1S^1 \xrightarrow{j_*} H_1(S^1, D^1_-) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}
\end{array}
\]

The calculation of $i_*$: $S^0 \to D^1$ can be done using naturality of the sum formula. Write $[\ast] \in H_0(\ast)$ for a fixed generator of this group (in singular homology, use the 0-chain defined by the identity map of the point). Then if $p_0, p_1: \ast \to S^0$ are inclusions of either point, then $p_0([\ast])$ and $p_1([\ast])$ are a basis of $H_0S^0$. They both map under $i_*$ to the same element of $H_0D^1$ since $ip_0, ip_1$ are homotopic maps $\ast \to D^1$. Both $ip_0$ and $ip_1$ are homotopy equivalences, so this element is the generator. So when I say $H_0S^0 \approx \mathbb{Z} \oplus \mathbb{Z}$ and $H_0D^1 \approx \mathbb{Z}$, these are the isomorphisms I have in mind.
On the top, we read off that $H_0(D^1_+, S^0) = 0$, while $H_1(D^1_+, S^0) \approx \mathbb{Z}$, and in fact is identified with the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(1, -1)$. Thus we have

\[
\begin{array}{ccccccc}
0 & \overset{i_*}{\longrightarrow} & 0 & \overset{j_*}{\longrightarrow} & \mathbb{Z} & \overset{j}{\longrightarrow} & \mathbb{Z} \oplus \mathbb{Z} & \overset{(a,b)\mapsto a+b}{\longrightarrow} & \mathbb{Z} & \overset{j_*}{\longrightarrow} & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \text{id} & & \downarrow & & \text{id} & & \downarrow & & \\
0 & \overset{i_*}{\longrightarrow} & H_1S^1 & \overset{j_*}{\longrightarrow} & \mathbb{Z} & \overset{0}{\longrightarrow} & H_0S^1 & \overset{j_*}{\longrightarrow} & 0 & \longrightarrow & 0 \\
\end{array}
\]

Thus $H_0(S^1) \approx \mathbb{Z}$ and $H_0(S^1) \approx 0$.

Going back up along the long exact sequences, for $q \geq 2$ we have

\[
\begin{array}{ccccccccc}
H_qS^0 & \overset{i_*}{\longrightarrow} & H_qD^1_+ & \overset{j_*}{\longrightarrow} & H_q(D^1_+, S^0) & \overset{\partial}{\longrightarrow} & H_{q-1}S^0 & \overset{i_*}{\longrightarrow} & H_{q-1}D^1_+ \\
\downarrow & & \downarrow & & f_* & & \sim & & \\
H_qD^1 & \overset{i_*}{\longrightarrow} & H_qS^1 & \overset{j_*}{\longrightarrow} & H_q(S^1, D^1_+) & \overset{\partial}{\longrightarrow} & H_{q-1}D^1 & \overset{i_*}{\longrightarrow} & H_{q-1}S^1 \\
\end{array}
\]

which is

\[
\begin{array}{ccccccccc}
0 & \overset{i_*}{\longrightarrow} & 0 & \overset{j_*}{\longrightarrow} & H_q(D^1_+, S^0) & \overset{\partial}{\longrightarrow} & 0 & \overset{i_*}{\longrightarrow} & 0 \\
\downarrow & & \downarrow & & f_* & & \sim & & \\
0 & \overset{i_*}{\longrightarrow} & H_qS^1 & \overset{j_*}{\longrightarrow} & H_q(S^1, D^1_+) & \overset{\partial}{\longrightarrow} & 0 & \overset{i_*}{\longrightarrow} & H_{q-1}S^1 \\
\end{array}
\]

which gives $H_qS^1 \approx H_q(S^1, D^1_+) \approx H_q(D^1_+, S^0) \approx 0$.

- When $n = 2$, use long exact sequences for $(D^2_+, S^1)$ and $(S^2, D^2_2)$.

\[
\begin{array}{ccccccccc}
H_1S^1 & \overset{i_*}{\longrightarrow} & H_1D^2_+ & \overset{j_*}{\longrightarrow} & H_1(D^2_+, S^1) & \overset{\partial}{\longrightarrow} & H_0S^1 & \overset{i_*}{\longrightarrow} & H_0D^2_+ & \overset{j_*}{\longrightarrow} & H_0(D^2_+, S^1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & f_* & & \sim & & \downarrow & & f_* & & \sim \\
H_1D^2 & \overset{i_*}{\longrightarrow} & H_1S^2 & \overset{j_*}{\longrightarrow} & H_1(S^2, D^2_2) & \overset{\partial}{\longrightarrow} & H_0D^2_+ & \overset{i_*}{\longrightarrow} & H_0S^2 & \overset{j_*}{\longrightarrow} & H_0(S^2, D^2_2) & \longrightarrow & 0 \\
\end{array}
\]

is

\[
\begin{array}{ccccccccc}
\mathbb{Z} & \longrightarrow & 0 & \overset{j_*}{\longrightarrow} & H_1(D^2_+, S^1) & \overset{\partial}{\longrightarrow} & \mathbb{Z} & \overset{\text{id}}{\longrightarrow} & \mathbb{Z} & \overset{j_*}{\longrightarrow} & H_0(D^2_+, S^1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & f_* & & \sim & & \downarrow & & f_* & & \sim \\
0 & \overset{i_*}{\longrightarrow} & H_1S^2 & \overset{j_*}{\longrightarrow} & H_1(S^2, D^2_2) & \overset{\partial}{\longrightarrow} & \mathbb{Z} & \overset{\text{id}}{\longrightarrow} & \mathbb{Z} & \overset{j_*}{\longrightarrow} & H_0(S^2, D^2_2) & \longrightarrow & 0 \\
\end{array}
\]

Note that $H_0S^1 \rightarrow H_0D^2$ is iso: use composite $S^1 \rightarrow D^2 \rightarrow \ast$, and note that both maps to $\ast$ induce iso on $H_0$. From this we read off $H_0S^2 \approx \mathbb{Z}$, and $H_1S^2 \approx 0$.

\[
\begin{array}{ccccccccc}
H_2S^1 & \overset{i_*}{\longrightarrow} & H_2D^2_+ & \overset{j_*}{\longrightarrow} & H_2(D^2_+, S^1) & \overset{\partial}{\longrightarrow} & H_1S^1 & \overset{i_*}{\longrightarrow} & H_1D^2_+ \\
\downarrow & & \downarrow & & f_* & & \sim & & \\
H_2D^2 & \overset{i_*}{\longrightarrow} & H_2S^2 & \overset{j_*}{\longrightarrow} & H_2(S^2, D^2_2) & \overset{\partial}{\longrightarrow} & H_1D^2_+ & \overset{i_*}{\longrightarrow} & H_1S^2 \\
\end{array}
\]
gives
\[
\begin{array}{cccc}
0 & \xrightarrow{i_*} & 0 & \xrightarrow{j_*} & H_2(D_+^2, S^1) & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{i_*} & 0 \\
0 & \xrightarrow{i_*} & H_2S^2 & \xrightarrow{j_*} & H_2(S^2, D_+^2) & \xrightarrow{\partial} & 0 & \xrightarrow{i_*} & 0 \\
\end{array}
\]
gives \(H_2S^2 \approx \mathbb{Z}\).

For \(q \geq 3\): the same idea shows \(H_qS^2 \approx H_{q-1}S^1 \approx 0\): every map in the zig-zag
\[
H_qS^2 \to H_q(S^2, D_-^2) \leftarrow H_q(D_+^2, S^1) \to H_qS^1
\]
is an isomorphism.

• When \(n \geq 2\), the same ideas give
\[
H_nS^n \approx \begin{cases} 
\mathbb{Z} & \text{if } q = 0, n, \\
0 & \text{else.}
\end{cases}
\]

In particular, for \(q \geq 2\), every map in
\[
H_qS^n \to H_q(S^n, D_-^n) \leftarrow H_q(D_+^n, S^{n-1}) \to H_qS^{n-1}
\]
is iso.

**Good pairs.** Say that \((X, A)\) is a good pair if \(A\) is closed and is a deformation retract of a neighborhood \(U\). That is, there is an open subset \(U\) in \(X\), and a homotopy \(H_t: U \to U\) such that \(H_0 = \text{id}_U\), \(H_1(U) \subseteq A\), and \(H_t|_A = \text{id}_A\). (I don’t think “good pair” is really standard terminology, but Hatcher uses it, so we will too.) The above argument shows that \(H_s(X, A) \to H_s(X, U)\) and \(H_s(X - A, U - A) \to H_s(X, U)\) are isomorphisms.

**Proposition.** Suppose \((X, A)\) is a good pair. Then the evident map \(H_s(X, A) \to H_s(X/A, A/A)\) is an isomorphism.

**Proof.** Let \(U\) be a neighborhood which deformation retracts to \(A\), and consider
\[
\begin{array}{cccc}
(X, A) & \xrightarrow{i} & (X, U) & \xleftarrow{\phantom{i}} & (X - A, U - A) \\
\downarrow & & \downarrow & & \downarrow \\
(X/A, A/A) & \xrightarrow{i} & (X/A, U/A) & \xleftarrow{\phantom{i}} & (X - A, U - A)
\end{array}
\]

Observe that \((X/A, A/A)\) is also a good pair, since \(U/A\) deformation retracts back to \(A/A\); also note that \((X/A - A/A, U/A - A/A) = (X - A, U - A)\). Thus the horizontal maps all induce isomorphisms in homology, and thus the vertical maps do too. □

**Example.** Since \((D^n, S^{n-1})\) is a good pair, \(H_*(D^n, S^{n-1}) \approx H_*(S^n, \text{pt})\).

**Remark.** The map \(H_*(X, A) \to H_*(X/A, A/A)\) can fail to be an isomorphism in general. For instance, let \(X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) \mid 0 < x \leq 1\}\), a compact topologists sine curve, and \(A = \{(0, y) \mid -1 \leq y \leq 1\}\). Then the projection map \(X \to I\) to the \(x\)-axis factors through a homeomorphism \(X/A \to I\). Since \(X\) is not path connected, we calculate
\[
H_0(X, A) \approx \mathbb{Z}, \quad H_0(X/A, A/A) = H_0(I, \{0\}) = 0.
\]
Reduced homology. Reduced homology is a slight modification of homology, so that the homology of a point is identically 0. There are actually two ways to think about this, depending on whether you are working with pointed or unpointed spaces. I'll give the more general one first.

Unpointed version of reduced homology. Define the augmented chain complex $\tilde{C}_*(X)$ of a space $X$ by $\tilde{C}_q(X) = C_q(X)$ if $q \neq -1$, and $\tilde{C}_{-1} = \mathbb{Z}$. Thus we have
\[
\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \to \cdots,
\]
where $\partial_0(\sigma) \overset{\text{def}}{=} 1$ for any $\sigma : \Delta^0 \to X$. Reduced homology is defined by $\tilde{H}_q(X) \overset{\text{def}}{=} H_q(\tilde{C}_*(X))$.

There is a natural map of chain complexes $\tilde{C}_*(X) \to C_*(X)$, inducing a map $\tilde{H}_*(X) \to H_*(X)$. This is part of a short exact sequence of chain complexes $0 \to \mathbb{Z}[-1] \to \tilde{C}_*(X) \to C_*(X) \to 0$, where $\mathbb{Z}[-1]$ is the complex with $\mathbb{Z}$ in degree $-1$, and is 0 elsewhere. Thus $\tilde{H}_q(X) \to H_q(X)$ an isomorphism for $q \neq 0, -1$, and an exact sequence
\[
0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\partial_0} \mathbb{Z} \to \tilde{H}_{-1}(X) \to 0.
\]

The boundary map sends the class of $\sum n_\sigma \sigma$ to $\sum n_\sigma$.

Given a point $x_0 \in X$, we have a class $[x_0] \in H_0(X)$ which projects to the generator under $H_0(X) \to \mathbb{Z}$, so this map is surjective and we get
\[
\tilde{H}_0(X) \oplus H_0(\{x_0\}) \xrightarrow{\sim} H_0(X), \quad \tilde{H}_{-1}(X) = 0.
\]

The direct sum decomposition of $H_0(X)$ depends on the choice of the point $x_0 \in X$. Since $X$ need not be pointed, it could be empty! In that case, $\tilde{H}_{-1}(\varnothing) = \mathbb{Z}$, while $\tilde{H}_0(\varnothing) = 0$ if $q \neq -1$.

We can derive properties for reduced homology analogous to the ones for non-reduced. Homotopy works exactly the same. The relative theory for reduced homology is not changed, so exactness has the form
\[
H_{n+1}(X, A) \xrightarrow{\partial} \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \xrightarrow{\partial} \tilde{H}_{n-1}(A),
\]
coming from the short exact sequence of complexes $0 \to \tilde{C}_*(A) \to \tilde{C}_*(X) \to C_*(X, A) \to 0$.

The dimension axiom can be replaced with $\tilde{H}_q(S^0) = 0$ if $q \neq 0, 1$, $\tilde{H}_0(S^0) = \mathbb{Z}$. Reduced homology does not satisfy the sum axiom.

Pointed version of reduced homology. Consider a space $X$ equipped with a basepoint $x_0$. There are maps $i : \{x_0\} \to X$ and $q : X \to \{x_0\}$, with $qi = \text{id}$. In the exact sequence
\[
\xrightarrow{q_*} H_k(\{x_0\}) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, x_0) \xrightarrow{q_*} H_k(X) \xrightarrow{i_*} H_k(X, x_0) \xrightarrow{q_*} H_k(X) \xrightarrow{i_*} H_k(X, x_0)
\]
since $q_* \circ i_* = \text{id}$, we obtain a direct sum decomposition
\[
(j_*, q_*): H_k(X) \xrightarrow{\sim} H_k(X, x_0) \oplus H_k(\text{pt}).
\]

Of course, this means $H_k(X) = H_k(X, x_0)$ if $k \neq 0$, and $H_0(X) \approx H_0(X, x_0) \oplus \mathbb{Z}$.

It is not hard to check that the composite map $\tilde{H}_0(X) \to H_0(X) \to H_0(X, x_0)$ is an isomorphism. Thus, if $(X, x_0)$ is a pointed space, we can identify $\tilde{H}_*(X)$ with $H_*(X, x_0)$.

Proposition. If $(X, A)$ is a good pair, then $H_*(X, A) \approx \tilde{H}_*(X/A)$.

If one is only working with pointed spaces and pointed maps, then it is usual simply to identify reduced homology with $H_*(X, x_0)$. 
Example. We have \( \tilde{H}_q(S^n) = 0 \) if \( n \neq q \), and \( \tilde{H}_n(S^n) = \mathbb{Z} \). This can be seen using the isomorphisms

\[
\tilde{H}_{q-1}(S^{n-1}) \cong H_q(D^n, S^{n-1}) \cong \tilde{H}_q(D^n / S^{n-1}) = \tilde{H}_q(S^n)
\]

which are now valid for all \( q \in \mathbb{Z} \), and all \( n \geq 0 \). This uses that \( \tilde{H}_*(D^n) \approx 0 \), and that \((D^n, S^{n-1})\) is a good pair.

We get a stronger version of the good pair result.

Proposition. Let \((X, A)\) be a good pair, and let \( f: A \to B \) be a map. Let \( Y = X \cup_A^f B \) be the pushout of the inclusion \( A \to X \) along \( f \). Then \((Y, B)\) is a good pair, and the induced map \( H_*(X, A) \to H_*(Y, B) \) is an isomorphism.

Proof. It is straightforward to see that \((Y, B)\) is also a good pair (using \( V = U \cup_A^f B \) and the obvious map gives a homeomorphism \( X/A \approx Y/B \), whence \( \tilde{H}_*(X/A) \approx \tilde{H}_*(Y/B) \), and thus \( H_*(X, A) \approx H_*(Y, B) \).

\( \square \)

Explicit chain representatives. I want to describe a candidate cycle which represents a generator of \( H_n(D^n, S^{n-1}) = \mathbb{Z} \). Using a homeomorphism \( D^n \approx \Delta^n \), we want a representative for the generator of \( H_n(\Delta^n, \partial \Delta^n) \approx \mathbb{Z} \). We have an obvious candidate, namely the identity map \( \iota_n: \Delta^n \to \Delta^n \) viewed as an element of \( C_n(\Delta^n, \partial \Delta^n) \).

Proposition. The element \( [\iota_n] \in H_n(\Delta^n, \partial \Delta^n) \) is a generator.

Because we have already computed the simplicial homology of this pair, i.e., the homology of \( \Delta_\ast(\Delta^n, \partial \Delta^n) \), this implies:

Proposition. The inclusion \( \Delta_\ast(\Delta^n, \partial \Delta^n) \to C_\ast(\Delta^n, \partial \Delta^n) \) induces an isomorphism in homology.

We do this by induction on \( n \). If \( n = 0 \), then \((\Delta^0, \partial \Delta^0) = (\text{pt}, \emptyset)\), and the result is clear.

Now suppose \( n > 0 \). Consider the inclusion \( d^0: \Delta^{n-1} \to \Delta^n \), and let \( \Lambda^{n-1} \subset \Delta^n \) denote the union of the \( d^i(\Delta^{n-1}) \) for \( i = 1, \ldots, n \). Then \( \Lambda^{n-1} \) is a \( \Delta \)-subcomplex of \( \Delta^n \), and we may also view \( \Delta_{n-1} = d^0(\Delta_{n-1}) \) as a subcomplex. The subspace \( \Lambda^{n-1} \) is a “cone” on the vertex \( v_0 \), so is contractible. We have that

\[
\partial \Delta^{n-1} = \Lambda^{n-1} \cap \Delta^{n-1} \quad \text{and} \quad \partial \Delta^n = \Lambda^{n-1} \cup_{\partial \Delta^{n-1}} \Delta^{n-1}.
\]

(Secretly, we are writing \( S^{n-1} = D^{n-1} \cup_{S^{n-2}} D^{n-1} \) again.)

Consider the sequence of maps

\[
H_n(\Delta^n, \partial \Delta^n) \xrightarrow{d^0} \tilde{H}_{n-1}(\partial \Delta^n) \xrightarrow{\iota_{n-1}^*} H_{n-1}(\partial \Delta^n, \Lambda^{n-1}) \xrightarrow{(d_0)_*} H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}).
\]

These are all isomorphisms in singular homology; the first two using the long exact sequence in reduced homology and the fact that \( \Lambda^{n-1} \) and \( \Delta^n \) are contractible, and \((d_0)_* \), using excision and the good pair property.

We want to show that under this chain of isomorphisms, \( [\iota_n] \in H_n(\Delta^n, \partial \Delta^n) \) corresponds to \( [\iota_{n-1}] \in H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}) \). In fact, on the level of cycles,

\[
\iota_n \mapsto_{\tau} (\sum (-1)^j d^j) \mapsto (d^0) = (d_0^\#(\iota_{n-1})) \mapsto (\iota_{n-1}).
\]

The effect of the relative boundary operator is computed by “lifting” the cycle \( \iota_n \in C_n(\Delta^n, \partial \Delta^n) \) to a the chain \( \iota_n \in \tilde{C}_n(\Delta^n) \) and taking its boundary, using the exact sequence

\[
0 \to \tilde{C}_\ast(\partial \Delta^n) \to \tilde{C}_\ast(\Delta^n) \to C(\Delta^n, \partial \Delta^n) \to 0.
\]
Equivalence of simplicial and singular homology.

**Theorem.** For any $\Delta$-complex pair $(X,A)$, the natural inclusion of chain complexes $\Delta_*(X,A) \to C_*(X,A)$ induces isomorphisms $H^\Delta_n(X,A) \to H_n(X,A)$.

We'll show $H^\Delta_n(X) \to H_n(X)$ is an isomorphism, then use long exact sequences and the 5-lemma to get the relative version. In is important note that, although simplicial homology is not a functor on continuous maps, it is functorial on inclusions of subcomplexes: for any map $f: A \to X$ which is an inclusion of a sub-$\Delta$-complex, the map $f_\#$ carries $\Delta_*(A) \subseteq C_*(A)$ into $\Delta_*(X) \subseteq C_*(X)$.

Let $X_n \subseteq X$ denote the union of all simplices of dimension at most $n$. The first step is to show that each $H^\Delta_n(X_n) \to H_n(X_n)$ is an isomorphism for each $n$, by induction on $n$. The base case is $X_{-1} = \emptyset$, for which both groups are 0.

Consider the map

$$(\sigma_n): \coprod \Delta^n \to X_n$$

which is the coproduct of all the characteristic maps of $n$-cells. I claim that this induces an isomorphism $H_*([\coprod \Delta^n, \coprod \partial \Delta^n]) \to H_*(X_n, X_{n-1})$. This follows from the fact that $([\coprod \Delta^n, \coprod \partial \Delta^n]$ is a good pair, and $X_n = X_{n-1} \cup [\coprod \Delta^n, \coprod \Delta^n]$. (Spell this out, using $U \subset X_n$ which deformation retracts to $X_{n-1}$.)

In the commutative square

$$
\begin{array}{c}
\bigoplus H^\Delta_*(\Delta^n, \partial \Delta^n) \\ \downarrow \\
\bigoplus H_*(\Delta^n, \partial \Delta^n)
\end{array} \quad \begin{array}{c}
\longrightarrow H^\Delta_*(\coprod \Delta^n, \coprod \partial \Delta^n) \\ \downarrow \\
H_*(\coprod \Delta^n, \coprod \partial \Delta^n)
\end{array} \quad \begin{array}{c}
\longrightarrow H^\Delta_*(X_n, X_{n-1}) \\ \downarrow \\
H_*(X_n, X_{n-1})
\end{array}
$$

all horizontal maps are isos, and the left vertical map is iso using the sum axiom for singular homology, and the isomorphism of the two homologies on $(\Delta^n, \partial \Delta^n)$. Therefore, the right vertical map is an isomorphism.

Recall the set up. For each $n$-skeleton $X_n$, we have a diagram of chain complexes

$$
\begin{array}{ccc}
0 & \longrightarrow & \Delta_*(X_{n-1}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C_*(X_{n-1})
\end{array} \quad \begin{array}{ccc}
\Delta_*(X_n) & \longrightarrow & \Delta_*(X_n, X_{n-1}) \\
\downarrow & & \downarrow \\
C_*(X_n) & \longrightarrow & C_*(X_n, X_{n-1})
\end{array} \quad \begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
$$

in which the rows are exact. This gives long exact sequences

$$
\begin{array}{c}
\longrightarrow H^\Delta_* (X_{n-1}) \\
\downarrow \\
\longrightarrow H^\Delta_* (X_n)
\end{array} \quad \begin{array}{c}
\longrightarrow H^\Delta_* (X_n, X_{n-1}) \\
\downarrow \rho \\
\longrightarrow H^\Delta_* (X_{n-1})
\end{array}
$$

The vertical maps are isomorphisms at the pairs by what I proved earlier, and at $X_{n-1}$ by induction, so they are isomorphisms at $X_n$ by the 5-lemma.

To prove it for $X = \bigcup X_n$, note that the obvious map $H^\Delta_n(X_n) \to H^\Delta_n(X)$ is an isomorphism if $n$ is large enough (e.g., $n \geq q + 1$). We'll be done if the obvious map $H_q(X_n) \to H_q(X)$ is an isomorphism if $n$ is large enough.

If $K \subseteq X$ is compact, then it meets only finitely many of the sets $\sigma_\alpha(\text{Int} \Delta^n)$. (We already proved this: if $K \subseteq X$ and $x_k \in K$ is a sequence of points each lying in a different open
We also get a long exact sequence if we replace homology with reduced homology.

The above observation about maps from compact spaces shows that any \( f \) is the identity map, which is impossible given the calculation of the homology groups.

**Mayer-Vietoris sequence.** This is really a restatement of excision.

**Proposition.** If \( X = \text{Int} A_1 \cup \text{Int} A_2 \) with \( B = A_1 \cap A_2 \), then there is a long exact sequence of the form

\[
H_{q+1} X \to H_q B \xrightarrow{((i_1)_*, -(i_2)_*)} H_q(A_1) \oplus H_q(A_2) \xrightarrow{((j_1)_*, (j_2)_*)} H_q(X) \to H_{q-1} B.
\]

We also get a long exact sequence if we replace homology with reduced homology.

**Proof.** The most direct proof given what we know is to use the exact sequence

\[
0 \to C_*(B) \xrightarrow{((i_1)_*, -(i_2)_*)} C_*(A_1) \oplus C_*(A_2) \xrightarrow{((j_1)_*, (j_2)_*)} C_*(X) \to 0,
\]

where \( U = \{A_1, A_2\} \).

Alternatively, you can prove it directly from excision, using the excision isomorphism \( H_*(A_2, B) \cong H_*(X, A_1) \) and a diagram chase; the boundary map in MV comes from

\[
H_{q+1} X \to H_{q+1}(X, A_1) \xrightarrow{\sim} H_{q+1}(A_2, B) \to H_q B.
\]

\[\square\]

**No retract theorem.**

**Proposition.** For \( n \geq 0 \), there is no map \( r: D^n \to S^{n-1} \) such that \( r|_{S^{n-1}}: S^{n-1} \to S^{n-1} \) is homotopic to the identity map.

**Proof.** Let \( i: S^{n-1} \to D^n \) be the inclusion. For \( n > 0 \), if \( r: D^n \to S^{n-1} \) exists such that \( r \circ i \sim \text{id} \), then the composite

\[
\tilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(S^{n-1})
\]

is the identity map, which is impossible given the calculation of the homology groups. \( \square \)

**Theorem (Brower fixed point theorem).** Every continuous map \( f: D^n \to D^n \) has a fixed point.

**Proof.** If \( f \) has no fixed point, then the map \( r: D^n \to S^{n-1} \) by \( r(x) = (x - f(x))/|x - f(x)| \) is such that \( g \circ i: S^{n-1} \to S^{n-1} \) is homotopic to the identity. \( \square \)

**Manifolds, and invariance of dimension.** The inclusion map \( D^n \to \mathbb{R}^n \) is a homotopy equivalence, and it restricts to a homotopy equivalence \( S^{n-1} \to \mathbb{R}^{n-1} \). Thus \( H_*(D^n, S^{n-1}) \rightarrow H_*(\mathbb{R}^n, \mathbb{R}^{n-1}) \) is an isomorphism.

**Theorem.** If \( m \neq n \), then \( \mathbb{R}^m \) and \( \mathbb{R}^n \) are not homeomorphic.

**Proof.** Let \( \phi: \mathbb{R}^m \to \mathbb{R}^n \) be a homeomorphism, and \( p \in \mathbb{R}^m \) any point. Then \( \phi \) induces an isomorphism \( H_*(\mathbb{R}^m, \mathbb{R}^m - \{p\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{\phi(p)\}) \), and thus an isomorphism \( H_*(D^m, S^{m-1}) \cong H_*(D^n, S^{n-1}) \), which is impossible unless \( m = n \). \( \square \)
Let $M$ be an $n$-dimensional topological manifold. Let $p \in M$, and let $U \subseteq M$ be an open Euclidean neighborhood of $p$. Then inclusion induces an isomorphism

$$Z \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_*(U, U \setminus \{p\}) \to H_*(M, M \setminus \{p\}).$$

To prove this, note that $M = (M \setminus \{p\}) \cup U$ is a union of open subsets, so we can apply excision. The group $H_n(M, M \setminus p)$ is occasionaly known as the local homology of $M$; we also may write $H_n(M[p])$ for this.

**Theorem** (Invariance of dimension). Suppose $M$ is an $m$-dimensional manifold, and $N$ is an $n$-dimensional manifold. If $M$ and $N$ are non-empty and homeomorphic, then $m = n$.

**Proof.** Consider local homology at any point. □

**Jordan curve theorem and generalizations.** The classical Jordan curve theorem says that if $C \subseteq \mathbb{R}^2$ is a subspace homeomorphic to a circle, then $\mathbb{R}^2 - C$ has exactly two connected components (which are also path components, $\mathbb{R}^2$ is locally path connected.) The proof will amount to showing that (for singular homology) $H_0(\mathbb{R}^2 - C) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Recall that a continuous map $f: X \to Y$ is an embedding if it induces a homeomorphism between $X$ and the subspace $f(X)$ of $Y$.

In the following, we are using the general version of reduced homology (for unpointed spaces).

**Complements of disk embeddings.** Before embedding into $\mathbb{R}^n$, we'll handle embeddings into $S^n$ first, which will contain the core of the proof. Before doing embedded spheres, we do embedded disks. Note that $D^k \approx I^k$.

**Proposition.** For any embedding $h: D^k \to S^n$, we have $\tilde{H}_q(S^n \setminus h(D^k)) = 0$ for all $q \in \mathbb{Z}$.

**Proof.** Induction on $k$, for fixed $n \geq 0$. If $k = 0$, then $S^n - h(D^0) \approx \mathbb{R}^n$, and we know $\tilde{H}_q(\mathbb{R}^n) = 0$. In general, we can replace the disk $D^k$ with a cube $I^k$.

Now suppose $k > 0$. Consider the decomposition

$$I^k = J_1 \cup J_2 := (I^{k-1} \times [0, \frac{1}{2}]) \cup (I^{k-1} \times [\frac{1}{2}, 1]),$$

and write $J_0 = J_1 \cap J_2 = I^{k-1} \times \{1/2\}$. Thus $J_1$ and $J_2$ are $k$-cubes, and $J_0$ a $(k - 1)$-cube.

Let $A_1 = S^n - h(J_1)$ and $A_2 = S^n - h(J_2)$. These are open subsets of the sphere, and

$$A_1 \cap A_2 = S^n - h(I^k), \quad A_1 \cup A_2 = S^n \setminus h(J_0).$$

Induction tells us that $\tilde{H}_*(A_1 \cup A_2) = 0$. The Mayer-Vietoris sequence gives a long exact sequence

$$\to \tilde{H}_{q+1}(A_1 \cup A_2) \to \tilde{H}_q(A_1 \cap A_2) \xrightarrow{(i_* j_*)} \tilde{H}_q(A_1) \oplus \tilde{H}_q(A_2) \to \tilde{H}_q(A_1 \cup A_2) \to$$

This reduces to isomorphisms

$$(i_* j_*): \tilde{H}_q(S^n \setminus h(I^k)) \to \tilde{H}_q(A_1) \oplus \tilde{H}_q(A_2).$$

Thus, a non-zero homology class in $S^n \setminus h(I^k)$ must project to a non-zero class in one of the $A_i$.

Now, suppose $\alpha \in \tilde{H}_q(S^n - h(I^k))$ is non-zero. By the above argument, we can choose a descending sequence of subintervals $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$ with $|I_j| = 1/2^j$, so that the image of $\alpha$ in each $\tilde{H}_q(S^n \setminus h(I^{k-1} \times I_j))$ is nonzero. Let $t$ be the unique element of $\bigcap I_j$. Thus

$$S^n - h(I^{k-1} \times \{t\}) = \bigcup_j (S^n - h(I^{k-1} \times I_j)),$$

where the right-hand side is a union of open subsets of $S^n$. 

By the induction hypothesis, \( \alpha \) must map to the 0 element in \( \tilde{H}_q(S^n \setminus h(I^{k-1} \times \{t\})) \). Let \( a \in \tilde{C}_q(S^n \setminus h(I^k)) \) be a cycle such that \( [a] = \alpha \), and let \( b \in \tilde{C}_{q+1}(S^n \setminus h(I^{k-1} \times \{t\})) \) be such that \( \partial(b) = a \). We have that \( b \) is a finite sum \( b = \sum n_i \sigma_i \), where each \( \sigma_i : \Delta^{q+1} \to (S^n \setminus h(I^{k-1} \times \{t\})) \). Since simplices are compact, there exists a \( j \) such that \( \sigma_i(\Delta^{q+1}) \subset S^n \setminus h(I^{k-1} \times I_j) \). Thus, \( b \in \tilde{C}_q(S^n \setminus h(I^{k-1} \times I_j)) \) for some \( j \), and therefore \( \alpha \) must restrict to 0 in \( \tilde{H}_q(S^n \setminus h(I^{k-1} \times I_j)) \), contradicting the construction. \( \square \)

Thus, removing any embedded disk from the sphere (of any dimension) gives a space with trivial reduced homology.

Recall what we just proved: \( \tilde{H}_*(S^n \setminus h(D^k)) \approx 0 \) for any \( n, k \).

Observation for later: The proof never actually assumed that \( k \leq n \). (Of course, there is no embedding when \( k > n \), which can be proved, for instance, by thinking about local homology.) Easy consequence: there is no homeomorphism \( D^k \approx S^n \) for any \( k, n \) (because \( \tilde{H}_{n-1}(\emptyset) \neq 0 \)).

**Corollary.** For any embedding \( h : D^k \to \mathbb{R}^n \), we have

\[
H_q(\mathbb{R}^n, \mathbb{R}^n \setminus h(D^k)) \approx \begin{cases} \mathbb{Z} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}
\]

That is, \( H_*(\mathbb{R}^n, \mathbb{R}^n \setminus h(D^k)) \approx H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \) for all \( k \).

**Proof.** We have isomorphisms \( H_*(\mathbb{R}^n, \mathbb{R}^n \setminus h(D^k)) \to H_*(S^n, S^n \setminus h(D^k)) \leftarrow \tilde{H}_*(S^n) \) by excision and LES. \( \square \)

**Homology of sphere complements in spheres.** Note that for the “usual” embedding \( S^k \subset \mathbb{R}^k \subset \mathbb{R}^n \subset S^n \), we have \( S^n \setminus S^k \approx S^{n-(k+1)} \text{Int} D^{k+1} \), which is homotopy equivalent to \( S^{n-k-1} \).

**Proposition.** For any embedding \( h : S^k \to S^n \), we have

\[
\tilde{H}_q(S^n \setminus h(S^k)) \approx \begin{cases} \mathbb{Z} & \text{if } q = n-k-1, \\ 0 & \text{otherwise.} \end{cases}
\]

That is, \( \tilde{H}_*(S^n \setminus h(S^k)) \approx \tilde{H}_*(S^{n-k-1}) \).

**Proof.** Induction on \( k \geq 0 \), with fixed \( n \geq 0 \). If \( k = 0 \), then \( S^n \setminus h(S^0) \approx \mathbb{R}^n \setminus \{p\} \), and we have computed this.

If \( k > 0 \), write \( S^k = D^k_+ \cup_{S^{k-1}} D^k_- \), and let \( U = S^n \setminus h(D^k_+) \) and \( V = S^n \setminus h(D^k_-) \). By the previous, the reduced homology of \( U \) and \( V \) vanishes, so

\[
\tilde{H}_q(S^n \setminus h(S^k)) \approx \tilde{H}_{q-1}(S^n \setminus h(S^{k-1}))
\]

for all \( q \in \mathbb{Z} \) by Mayer-Vietoris. \( \square \)

In brief: the complement of an embedded \( k \)-sphere in an \( n \)-sphere has the homology of a \( k \)-sphere, where \( k + \ell + 1 = n \).

**Corollary.** Any embedding \( h : S^n \to S^n \) is a homeomorphism. There are no embeddings \( h : S^k \to S^n \) with \( k > n \).

**Proof.** We have \( \tilde{H}_*(S^n \setminus h(S^n)) \) is a \( \mathbb{Z} \) in \( * = -1 \). The only space with this reduced homology is the empty space, so \( h \) is surjective.

For \( k > n \), the calculation of \( \tilde{H}_* \) of the complement is non-trivial in \( * \leq -2 \), which is impossible. \( \square \)
For an embedding \( h: S^k \to \mathbb{R}^n \subset S^n \), let \( U = S^n \setminus h(S^k) \) and \( V = \mathbb{R}^n \setminus h(S^k) \). Note that \( V \) is an open subset of the manifold \( U \) obtained by removing a single point \( p \) from \( U \): thus, excision applies to the pair \((U, V) = (U, U \setminus p)\). We have

\[
\bar{H}_*(U) \approx \begin{cases} 
\mathbb{Z} & \text{if } * = n - k - 1, \\
0 & \text{otherwise},
\end{cases}
\]

\[
H_*(U, V) \approx \begin{cases} 
\mathbb{Z} & \text{if } * = n, \\
0 & \text{otherwise},
\end{cases}
\]

so each is non-zero in exactly one dimension \((n - k - 1 \text{ and } n)\). Since \( k \geq 0, n \neq n - k - 1, \) so \( \bar{H}_*U \to H_*(U, V) \) is the 0 map in every degree. By the LES, \( \bar{H}_*V = \mathbb{Z} \oplus \mathbb{Z} \), where the \( \mathbb{Z}s \) are located in \(* = n - k - 1 \text{ and } * = n - 1\). Note that these are the \textit{same} dimension if \( k = 0 \) (homology of \( \mathbb{R}^n \setminus \{p, q\} \)). These contribute to \( \bar{H}_0V \) if either \( n = 1 \) or \( k = n - 1 \).

\textbf{Corollary} (Generalized Jordan Curve theorem). \textit{If} \( h: S^{n-1} \to \mathbb{R}^n \) \textit{is an embedding and} \( n \geq 2 \), \textit{then} \( \mathbb{R}^n \setminus h(S^{n-1}) \) \textit{has exactly two connected components; one has the homology of a point, the other has the homology of} \( S^{n-1} \).

\textit{Proof.} We have \( \bar{H}^0(\mathbb{R}^n \setminus h(S^{n-1}) \approx \mathbb{Z} \), so \( H^0 \approx \mathbb{Z} \oplus \mathbb{Z} \). Thus the complement has two path components, which are also connected components since \( \mathbb{R}^n \) is LPC. We also have \( H^{n-1} \approx \mathbb{Z} \).

\textit{Example.} \( U := S^n \setminus h(S^0) \) has the homology of \( S^{n-1} \). In this case, \( U \) is homeomorphic to \( \mathbb{R}^n \setminus 0 \).

\textit{Example.} \( U = S^3 \setminus h(S^1) \) has the homology of \( S^1 \). An embedded circle \( K \subset S^3 \) is a \textbf{knot}, and \( U \) is the \textbf{knot complement}.

However, \( U \) need not be homotopy equivalent to \( S^1 \). In fact, the fundamental group \( G := \pi_1U \) is a potentially non-trivial invariant of the knot. For instance, for the unknot, \( G \approx \mathbb{Z} \), but trefoil knot group is different.

Knot groups (for tame knots) can be computed using the “Wirtinger presentation”. For a knot presented by a knot diagram, \( K \approx (x_1, \ldots, x_n | r_1, \ldots, r_n) \), with a generator for each connected arc in the knot diagram, and a relation for each crossing. For the trefoil knot, \( G = \langle x_1, x_2, x_3 \ | \ x_1 x_2 x_1^{-1} = x_3, x_2 x_3 x_2^{-1} = x_1, x_3 x_1 x_3^{-1} = x_2 \rangle \approx \langle x_1, x_2 \ | \ x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \).

There is a surjective homomorphism \( G \to \Sigma_3 \) to the symmetric group (by \( x_1 \mapsto (12), x_2 \mapsto (23) \)). Thus \( G \) is not abelian, so not isomorphic to \( \mathbb{Z} \).

Schönflies theorem (all embeddings \( S^1 \to \mathbb{R}^2 \) are equivalent, up to homeomorphism of \( \mathbb{R}^2 \)). This implies that the bounded component of \( \mathbb{R}^2 \setminus C \) is always homeomorphic to an open disk.

Alexander horned sphere \( (\mathbb{R}^3 \setminus h(D^3) \) need not be simply connected). In this case, the space \( S^3 \setminus h(D^3) \) is also not simply connected, yet \( \bar{H}_*(S^3 - h(D^3)) = 0 \).

Knots (all embeddings \( S^1 \to \mathbb{R}^3 \) are not equivalent).

\textbf{Degree.} For a map \( f: S^n \to S^n \) (with \( n > 0 \)), the \textbf{degree} of \( f \) is the unique integer \( d \) such that \( f_*(\alpha) = d \alpha \) for \( \alpha \in H_n S^n \approx \mathbb{Z} \). We write \( \deg(f) = d \).

- The identity map has degree 1.
- Homotopic maps have the same degree.
- A non-surjective map has degree 0.
- \( \deg(gf) = \deg(g) \deg(f) \).
- If \( g \) is a homeomorphism, \( \deg(gfg^{-1}) = \deg(f) \).

\textbf{Proposition.} Let \( f: S^1 \to S^1 \) by \( f(z) = z^n \). Then \( \deg(f) = n \).
Proof. There are many ways to prove this. I’ll sketch one way, using explicit cycles.

Let \( q: I \to S^1 \) be \( q(t) = e^{2\pi it} \), inducing a homeomorphism \( I/\partial I \cong S^1 \). Given a chain \( a = \sum n_i \alpha_i \in C_1 I \) such that \( \partial(a) = (1) - (0) \in C_0 I \), we have that \([a]\) is the standard generator of \( H_1(I, \partial I) \), and so \([q_\#(a)]\) is the standard generator of \( H_1S^1 \). (This is because of the sequence of isomorphisms \( H_0(\partial I) \leftarrow H_1(I, \partial I) \to H_1(S^1, \{1\}) \leftarrow H_1S^1 \).)

For instance, let \( \alpha: 0 \to 1 \) and \( \beta: 1 \to 0 \) be constant velocity paths in \( I \). Then both \( q_\#(\alpha) \) and \( q_\#(\beta) \) represent the generator of \( H_1S^1 \). If \( f(z) = z^{-1} \), then clearly \( f_\#(q_\#(\alpha)) = f_\#(q_\#(\beta)) \), so \( \deg(f) = -1 \).

Similarly, for \( f(z) = z^n \) with \( n > 0 \), use \( a = \sum_{k=1}^n \alpha_k \), where \( \alpha_k : (k-1)/n \sim k/n \) is constant velocity. \( \square \)

Suspension. Given a space \( X \), the suspension of \( X \) is the quotient \( SX = (X \times D^1) \cup \{*, \sim \} \), where we identify \((x, -1) \sim *_- \) and \((x, 1) \sim *_+ \) for all \( x \in X \). We can think of \( S(X) \) as a union of two cones: \( S(X) = C_+(X) \cup_X C_-(X) \), where both \( C_\pm(X) \) are copies of the cone \( C(X) := (X \times I) \cup \{\} \sim (x, 1) \sim * \) is always contractible. Glue the two cones along the common copy of \( X = X \times \{0\} \subseteq C(X) \).

Exercise. What is \( S(\emptyset) \)?

Note that \( S(S^n) \approx S^{n+1} \). The pair \((C_+, X, X)\) is a good pair, which means that the zig-zag of maps

\[
\tilde{H}_{q+1}SX \to H_{q+1}(SX, C-X) \leftarrow H_{q+1}(C_+, X, X) \xrightarrow{\partial} \tilde{H}_qX
\]

are all isomorphisms. The induced map \( \sigma: \tilde{H}_qX \to \tilde{H}_{q+1}S(X) \) is called the suspension isomorphism.

If \( f: X \to Y \) is a map of spaces, then the diagram

\[
\begin{array}{ccc}
\tilde{H}_{q+1}SX & \xrightarrow{(sf)_*} & H_{q+1}(SX, C-X) & \xleftarrow{(C+f)_*} & H_{q+1}(C_+, X, X) & \xrightarrow{f_*} & \tilde{H}_qX \\
\tilde{H}_{q+1}SY & \xrightarrow{(sf,C-f)_*} & H_{q+1}(SY, C-Y) & \xleftarrow{(C+f,Y)_*} & H_{q+1}(C_+, Y, Y) & \xrightarrow{f_*} & \tilde{H}_qY \\
\end{array}
\]

commutes. That is, the zig-zag, and thus the suspension isomorphism, is natural with respect to continuous maps.

Applied to the case of \( X = Y = S^n \) and \( q = n \), we get

Proposition. If \( f: S^n \to S^n \), then \( \deg(S(f)) = \deg(f) \).

Proposition. If \( f: S^n \to S^n \) is a reflection through a hyperplane, then \( \deg(f) = -1 \). If \( f: S^n \to S^n \) is the antipodal map \( f(x) = -x \), then \( \deg(f) = (-1)^{n+1} \).

Proof. A reflection is homeomorphic to an iterated suspension of \( f: S^1 \to S^1 \) by \( f(z) = z^{-1} \). The antipodal map on \( S^n \) is a composite of \( n + 1 \) hyperplane reflections. \( \square \)

Proposition. If \( f: S^n \to S^n \) has no fixed point, then \( f \) is homotopic to the antipodal map, and so \( \deg(f) = (-1)^{n+1} \).

Thus, if \( \deg(f) \neq (-1)^{n+1} \), then \( f \) has a fixed point.

Proof. Let \( g_t(x) = ((1-t)f(x) - tx)/(1-t)f(x) - tx \) be a homotopy of maps \( S^n \to S^n \) between \( f \) and the antipodal map. This is well-defined, since the line segment connecting \( f(x) \) and \( -x \) cannot pass through the origin, exactly because \( f \) has no fixed point. \( \square \)

In particular, any fixed-point free self-map of a sphere is a homotopy equivalence.

Corollary. If \( G \) is a finite group acting freely on \( S^n \) with \( n \) even, then \( |G| = 1 \) or 2.
Proof. “Freely” means non-identity elements have no fixed points. If $G$ acts on $S^n$, there is a homomorphism $d: G \to \{\pm 1\}$ by $g \mapsto \deg(g)$. If $g \in G$ acts on $S^n$ with no fixed point, then $\deg(g) = (-1)^{n+1} = -1$ (since $n$ is even). If $G$ acts freely, then $g$ has fixed points only if its the identity element. Putting these together, we see that Ker $\deg = \{1\}$, so $\deg: G \to \{\pm 1\}$ is injective, so $|G| \leq 2$.

**Proposition.** If $n$ is even and $f: S^n \to S^n$ is a map, then there exists a point $x \in S^n$ such that $f(x) \in \{\pm x\}$.

Proof. If not, then we can construct homotopies $x \sim f(x) \sim -x$, by $t \mapsto ((1-t)f(x) \pm tx)/|(1-t)f(x) \pm tx|$. □

A **vector field** on $S^n$ is a continuous map $v: S^n \to \mathbb{R}^{n+1}$ such that $x \cdot v(x) = 0$. (In the language of differential geometry, it’s a “continuous section of the tangent bundle”.)

**Proposition.** If $n$ is even, there does not exist a nowhere vanishing vector field on $S^n$.

Proof. Let $v: S^n \to \mathbb{R}^{n+1}$ be a nowhere vanishing vector field, and let $f: S^n \to S^n$ by $f(x) = v(x)/|v(x)|$. Then since $x \cdot v(x) = 0$, we clearly have $f(x) \notin \{\pm x\}$ for all $x$. □

**More on vector fields.** If $n$ is odd, there does exist a nowhere vanishing vector field on $S^n$. Write $n = 2k - 1$. To produce a solution, it is enough to give an orthogonal $2k \times 2k$ matrix $A$ such that $A^2 = -I$. Then

$$x \cdot Ax = Ax \cdot A^2x = -Ax \cdot x,$$

so $x \cdot Ax = 0$ for all $x \in \mathbb{R}^{2k}$. For instance, if we write $\mathbb{R}^{2k} = \mathbb{C}^k$, then multiplication by $i$ supplies such a tranformation.

**Vector field problem.** How many linearly independent nowhere vanishing vector fields can be put on $S^n$? Let $\rho(n)$ be maximal number of such fields on $S^{n-1}$. Then $\rho(n)$ only depends on the number of powers of 2 in $n$. In fact, if $n = 2^d u$ with $u$ odd then

$$\rho(n) = 2d + \epsilon(d),$$

where $\epsilon(d)$ only depends on $d$ modulo 4, with $\epsilon(d) = 2^d - 1 - 2d$ for $d = 0, 1, 2, 3$. This many vector fields were constructed by Radon and Hurwitz, and Adams proved this is best possible, using $K$-theory.

(The Radon-Hurwitz method is to produce a collection of $n \times n$ orthogonal matricies $A_1, \ldots, A_d$ such that $A_d = -I$ and $A_i A_j + A_j A_i = 0$ for $i \neq j$. Then the maps $v_i : S^{n-1} \to S^{n-1}$ by $v_i(x) = A_i x$ give pairwise perpendicular vector fields. This can be formalized by the theory of Clifford algebras.)

Note that $\rho(n) = n - 1$ only if $n = 1, 2, 4, 8$. Thus the only parallelizable spheres are $S^0$, $S^1$, $S^3$, and $S^7$.

**Local degree.** A rough idea is that a map $f$ is degree $d$ if it is “$d$-to-one”, i.e., if the fibers of $f$ have size $d$. This doesn’t really work, but a better idea is that the $d$ is the size of the fibers of $f$, counted with multiplicity.

Given any neighborhood $U$ of a point $p \in S^n$, there are isomorphisms

$$\mathbb{Z} \approx H_n(S^n) \to H_n(S^n, S^n - p) \to H_n(U, U - p).$$

Thus, we have a standard way to identify such groups with $\mathbb{Z}$. In particular, there is a “canonical” generator of $H_n(U, U - p)$. I’ll write $[S^n] \in H_n(S^n)$ for the standard generator, and I’ll write $[U[p]]$ for its image in $H_n(U, U - p)$.

Now suppose that $p, q \in S^n$ such that $q = f(p)$, and suppose that there are no other points of $f^{-1}(q)$ near $p$; that is, there exists $U$ a neighborhood of $p$ and $V$ a neighborhood of $q$ such that $f(U - p) \subset (V - q)$. The **local degree** of $f$ at $p$ is then defined to be the integer
deg \ f \mid p = d$ describing the map 

$$\mathbb{Z} \approx H_n(U, U \setminus p) \xrightarrow{f_*} H_n(V, V \setminus q) \approx \mathbb{Z},$$

so that $f_*(\{p\}) = d \{q\}$. (Note that this doesn’t depend on the neighborhood $U$.)

**Example.** Let $g = S(f) : S^2 \to S^2$, where $f : S^1 \to S^1$ by $f(z) = z^n$ with $n \neq 0$; think of this as wrapping the sphere around itself $n$ times, with north and south poles at fixed points. The local degree at each pole is $n$, while the local degree at other points is 1 or $-1$ (depending on the sign of $n$).

Suppose that $f : S^n \to S^n$, and that $q \in S^n$ such that $f^{-1}(q)$ is a finite set $\{p_1, \ldots, p_m\}$. This implies that we can define $\text{deg } f \mid p_k$ for each $k$. The local degree formula says that

$$\text{deg}(f) = \sum_k \text{deg } f \mid p_k.$$ 

Let $V$ be a neighborhood of $q$, and let $U_1, \ldots, U_m$ be disjoint neighborhoods of $p_1, \ldots, p_m$ such that $f(U_k) \subseteq V$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\coprod (U_k \setminus p_k) & \to & V \setminus q \\
\downarrow & & \downarrow \\
\coprod U_k & \to & V \\
\downarrow & & \downarrow \\
S^n & \to & S^n \\
\end{array}
$$

since $f^{-1}(q) = \{p_1, \ldots, p_m\}$, so $U_k \cap f^{-1}(q) = \{p_k\}$. This implies that $f(U_k \setminus p_k) \subseteq (V \setminus q)$. Consider

$$
\begin{array}{ccc}
\bigoplus H_n(U_k, U_k \setminus p_k) & \xrightarrow{\sum} & H_n(\coprod U_k, \coprod (U_k \setminus p_k)) \\
\downarrow \sim & & \downarrow \sim \\
H_n(S^n, S^n \setminus p_k) & \xleftarrow{\sim} & H_n(S^n, S^n \setminus f^{-1}(q)) \\
\text{EXCISION} & & \text{EXCISION} \\
\downarrow g_* & & \downarrow L.E.S \\
H_n(S^n) & \xrightarrow{f_*=d_\ast} & H_n(S^n) \\
\end{array}
$$

The horizontal maps in the right-hand column are induced by $f$; all other maps are the “obvious” ones. The left-hand square commutes by naturality: for each $k$, the composite 

$$(U_k, U_k \setminus p_k) \to \coprod (U_k, U_k \setminus p_k) \to (S^n, S^n \setminus f^{-1}(q)) \to (S^n, S^n \setminus p_k)$$

is the obvious inclusion $i_k$.

On the top, $[U_k, p_k] \to d_k [V, q]$, where $d_k = \text{deg } f \mid p_k$. On the bottom, $[S^n] \to d[S^n]$, where $d = \text{deg } f$. Basically, we need to know that $g_*([S^n]) = \sum(i_k)_*([U_k, p_k])$, where $i_k : U_k \to S^n$ are the inclusions. This is shown by composing with the maps induced by $(S^n, S^n \setminus f^{-1}(q)) \to (S^n, S^n \setminus p_k)$, and using naturality: for each $k$, $g_*[S^n] \to [S^n|p_k]$, while $(i_k)_*[U_k|p_k] = [S^n|p_k]$. 


Remark. We'll show later that any compact, connected, and orientable manifold $M$ of dimension $n$ has $H_n M \approx \mathbb{Z}$; a choice of generator $[M] \in H_n M$ corresponds exactly to an orientation of $M$.

Given two compact, connected, and oriented $n$-manifolds, you can define the notion of the degree of a map $f: M \to N$ in exactly the same way: $f_* [M] = (\deg f) [N]$. There is an analogous local degree formula.

**Cellular homology.** Let $X$ be a CW complex. Then $X = \bigcup X_n$, where $X_n$ is the union of all cells of dimension at most $n$. We will define a chain complex $C^\bullet_{CW}(X)$, which depends on the CW structure.

The groups are defined to be

$$C^\bullet_{CW}(X) \overset{\text{def}}{=} H_n(X_n, X_{n-1}).$$

Remember that $X_n$ is obtained from $X_{n-1}$ by attaching cells along. Thus $X_n$ is a pushout of $\coprod S^{n-1}$ along a map $(\phi _n): \coprod S^{n-1} \to X_{n-1}$. Since the cell inclusion is a good pair, excision gives

$$H_q(X_n, X_{n-1}) = \bigoplus H_q(D^n, S^{n-1}).$$

Thus $C^\bullet_{CW}(X)$ is a free abelian group on the $n$-cells of the CW-structure. Write $e_\alpha = \Phi_\phi (\iota_n)$, where $\iota_n$ is the “standard” generator of $H_n(D^n, S^{n-1})$; then the $e_\alpha$’s are a basis of $C^\bullet_{CW}(X)$.

The boundary map $\partial_{CW}: C^n_{CW}(X) \to C_{n-1}^{CW}(X)$ is defined by

$$H_n(X_n, X_{n-1}) \overset{\partial}{\to} H_n(X_{n-1}) \overset{j_*}{\to} H_n(X_{n-1}, X_{n-2}),$$

using the boundary map for $(X_n, X_{n-1})$. Check that $\partial_{CW} \partial_{CW} = 0$.

**Exercise.** Check that if $X$ is a $\Delta$-complex, then $C^\bullet_{CW}(X) \approx \Delta_\bullet(X)$. (It’s clear that $C^\bullet_{CW}(X) \approx \Delta_\bullet(X)$; the main point is to make sure that the formulas for the boundary maps are the same.)

**Proposition.** $H^\bullet_{CW}(X) \approx H_\bullet(X)$.

**Warning.** The assignment $X \mapsto C^\bullet_{CW}(X)$ of cellular chains is not a functor on the category of CW-complexes and continuous maps. Thus it is not obvious that $H^\bullet_{CW}$ is a functor, though it is by the proposition.

However, $C^\bullet_{CW}$ is functorial with respect to cellular maps, i.e., maps $f: X \to Y$ between CW-complexes such that $f(X_k) \subseteq Y_k$ for all $k$.

**Remark.** The proof I’m going to give is similar to the proof that simplicial homology of a $\Delta$-complex computes singular homology, but there is a difference. For a $\Delta$-complex, there is an inclusion $\Delta_\bullet(X) \subset C_\bullet(X)$ of chain complexes, and thus a natural map $H^\bullet_\Delta(X) \to H_\bullet(X)$, which we showed is an isomorphism.

The cellular chain complex $C^\bullet_{CW}(X)$ cannot generally be identified with a subcomplex of singular chains. So there is no a priori map $H^\bullet_{CW}(X) \to H_\bullet(X)$ to prove is an isomorphism.

**Proof.** First let’s do the case when $X$ is finite dimensional, so $X = X_m$. Thus $H_q(X) = H_q(X_m)$. Also, note that $H_q(X_{-1}) = H_q(\emptyset) = 0$.

Recall that attaching an $n$-cell to a space only changes its homology in dimensions $n$ and $n-1$. Thus, for a given $q$, we must have

$$0 = H_q X_{-1} \approx H_q X_0 \approx \cdots \approx H_q X_{q-1}, \quad H_q X_{q+1} \approx H_q X_{q+2} \approx \cdots \approx H_q X_m = H_q X.$$

To examine what happens in $H_q X_q$ and $H_q X_{q+1}$, we will need to use the exact sequences

$$0 \to H_q X_q \overset{j_*}{\to} H_q(X_q, X_{q-1}) \overset{\partial}{\to} H_{q-1} X_{q-1} \overset{i_*}{\to} H_{q-1} X_q \to 0.$$
and
\[ 0 \to H_{q+1}X_{q+1} \xrightarrow{j_*} H_{q+1}(X_{q+1}, X_q) \xrightarrow{\partial} H_qX_q \xrightarrow{i_*} H_qX_{q+1} \to 0. \]
Note that the maps marked “\( j_* \)” here are both injective.

The exact sequences show that
\[ H_q(X_q) \approx \text{Ker}[H_q(X_q, X_{q-1}) \xrightarrow{\partial} H_{q-1}(X_{q-1})], \]
\[ H_q(X_{q+1}) \approx H_q(X_q)/\text{Im}[H_{q+1}(X_{q+1}, X_q) \xrightarrow{\partial} H_q(X_q)]. \]
I claim that \( H_q(X_q) \approx Z_q^{CW}(X) \), and that \( H_q(X_{q+1}) \approx Z_q^{CW}(X)/B_q^{CW}(X) \).

Consider the composite
\[ H_q(X_q) \xrightarrow{j_*} H_q(X_q, X_{q-1}) \xrightarrow{\partial} H_{q-1}(X_{q-1}) \xrightarrow{j_*} H_{q-1}(X_{q-1}, X_{q-2}) \]
Since the second map marked \( j_* \) is injective, \( \text{Ker}(\partial) = \text{Ker}(\partial_{CW}) = Z_q^{CW}(X) \). Since the sequence is exact at \( H_q(X_q, X_{q-1}) \) and the first \( j_* \) is injective, we have \( H_q(X_q) \approx \text{Im}(j_*) = \text{Ker}(\partial) = Z_q^{CW}(X) \).

In
\[ H_{q+1}(X_{q+1}, X_q) \xrightarrow{\partial} H_q(X_q) \xrightarrow{j_*} H_q(X_q, X_{q-1}) \]
since \( j_* \) is injective, \( B_q^{CW}(X) = \text{Im}(\partial_{CW}) = j_*(\text{Im}(\partial)) \subseteq Z_q^{CW}(X) \). Thus \( H_q(X_{q+1}) = H_q(X_q)/\text{Im}(\partial) \approx Z_q^{CW}(X)/B_q^{CW}(X) \).

If \( X \) is infinite dimensional, it is enough to show that \( H_q(X_m) = H_q(X) \) is an isomorphism for \( m > q \), since it is clear that \( H_q^{CW}(X_m) = H_q^{CW}(X) \) is iso in this range. These groups stabilize, so it suffices to show that \( H_q(X) = \text{lim}_m H_q(X_m) \), which follows using a compactness argument. \( \square \)

**Degree formulas.** The boundary map \( \partial_{CW}: C_1^{CW}(X) \to C_0^{CW}(X) \) is easy to describe: \( \partial_{CW}(e_\alpha) = e_1 - e_0 \), where \( e_1 \) and \( e_0 \) are the cells at the endpoints. For higher dimensions, we have the following formula. If \( e_\alpha \) is a generator of \( C_n^{CW}(X) \), then
\[ \partial_{CW}(e_\alpha) = \sum_{\beta} d_{\alpha\beta} e_\beta, \]
where \( d_{\alpha\beta} \) is the degree of \( S^{n-1} \xrightarrow{\phi_\alpha} X_{n-1} \xrightarrow{\psi_\beta} S^{n-1} \), where
\[ \psi_\beta: X_{n-1} \to X_{n-1}/(X_{n-1} - \Phi_\beta(\text{Int} D^{n-1})) \xhookrightarrow{\sim} D^{n-1}/S^{n-2} \xrightarrow{q} S^{n-1} \]
is the “pinch” map.

**Proof.** (Skip this proof in class.) Recall that \( C_n^{CW}(X) \) is a free abelian group on the set of \( n \)-cells of the CW-structure. The idea is that the generator \( e_\alpha \in C_n^{CW}(X) \) is the image of the generator under \( (\Phi_\alpha)_*: H_n(D^n, S^{n-1}) \to H_n(X_n, X_{n-1}) \). We need to think a little bit more carefully about this.

For each \( n \) we have chosen a “standard” generator \( t_n \in H_n(D^n, S^{n-1}) \). I want these to be consistent with each other, in the sense that the sequence of isomorphisms
\[ H_{n-1}(D^{n-1}, S^{n-2}) \xrightarrow{\pi_*} H_{n-1}(D^{n-1}/S^{n-2}) \xrightarrow{q_*} H_{n-1}(S^{n-1}) \xrightarrow{d} H_n(D^n, S^{n-1}) \]
relates \( t_{n-1} \) and \( t_n \); i.e., \( q_*(\pi_*(t_{n-1})) = \partial(t_n) \). This sequence implicitly uses a choice of homeomorphism \( q: D^n/S^{n-1} \to S^n \) (think of this as choosing a CW-structure for each sphere). Fixing these choices fixes the choice of \( t_n \) (where \( t_0 \in H_0(D^0, \emptyset) \) is the obvious generator).
Also, note that the homeomorphism \( q \) was used in the definition of the pinch maps \( \psi_{\alpha} : X_n \to S^n \). (The “natural” map is \( X_n \to D^n / S^{n-1} \); the identification of the latter with \( S^n \) has to be chosen.)

The composite

\[
(D^n, S^{n-1}) \xrightarrow{\Phi_{\alpha}} (X_n, X_{n-1}) \xrightarrow{\psi_{\alpha}} (S^n, *)
\]

is composite \( q \circ \pi \) of the canonical quotient map \( \pi \) with \( q : D^n / S^{n-1} \to S^n \). On the other hand, if \( \alpha \neq \beta \), then the composite \( \psi_{\beta} \circ \phi_{\alpha} \) is the constant map. Thus, in homology, if we set \( e_{\alpha} = (\Phi_{\alpha})_{*}(t_n) \in H_n(X_n, X_{n-1}) \), we have that

\[
(\psi_{\beta})_{*}(e_{\alpha}) = \begin{cases} q_{*}(\pi_{*}(t_n)) = \partial(t_{n+1}) & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}
\]

To compute the boundary map, write \( \partial_{\text{CW}}(e_{\alpha}) = \sum d_{\alpha\beta} e_{\beta} \) where \( d_{\alpha\beta} \in \mathbb{Z} \). Compute that

\[
(\psi_{\beta})_{*}(\partial_{\text{CW}}(e_{\alpha})) = d_{\alpha\beta} \partial(t_n) \in H_{n-1}(S^{n-1}).
\]

The commutative diagram

\[
\begin{array}{ccc}
H_n(D^n, S^{n-1}) & \xrightarrow{\partial} & H_{n-1}(S^{n-1}) \\
(\Phi_{\alpha})_* & \sim & (\phi_{\alpha})_* \\
H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow{j_*} & H_{n-1}(X_{n-1}, X_{n-2}) \\
(\psi_{\beta})_* & \sim & (\psi_{\beta})_* \\
H_{n-1}(S^{n-1}) & \sim & H_{n-1}(S^{n-1}, *)
\end{array}
\]

shows that \( (\psi_{\beta})_{*}(\phi_{\alpha})_{*}(\partial(t_n)) = d_{\alpha\beta} \partial(t_n) \), and thus \( d_{\alpha\beta} = \deg(\psi_{\beta} \circ \phi_{\alpha}) \).

(The key here was to make sure that we use the same generator of \( H_{n-1}(S^{n-1}) \) in both source and target of \( \psi_{\beta} \circ \phi_{\alpha} \). Otherwise, our formula might be off by a sign.)

If \( f : X \to Y \) is a cellular map between CW-complexes, i.e., \( f(X_n) \subseteq Y_n \), then there is an induced map \( f_* : C^*_{\text{CW}}(X) \to C^*_{\text{CW}}(Y) \), and we have the formula

\[
f_*(e_{\alpha}) = \sum_{\beta} f_{\alpha\beta} e_{\beta},
\]

where \( f_{\alpha\beta} \) is the degree of \( S^n \approx D^n / S^{n-1} \xrightarrow{\Phi_{\alpha}} X_n / X_{n-1} \xrightarrow{f} Y_n / Y_{n-1} \xrightarrow{\psi_{\beta}} S^n \).

**Surfaces.** Recall the “polyhedral” CW-structures on \( M_g \) and \( N_k \). The complex \( C^*_{\text{CW}}M_g \) is isomorphic to

\[
\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}.
\]

The cellular chain groups are then also the homology groups.

The complex \( C^*_{\text{CW}}N_k \) is isomorphic to

\[
\mathbb{Z} \xrightarrow{1+(2, \ldots, 2)} \mathbb{Z}^k \xrightarrow{0} \mathbb{Z}.
\]

In this case, we see that \( H_0 = \mathbb{Z}, H_2 = 0 \), while

\[
H_1 \approx \mathbb{Z}^k / \langle (2, \ldots, 2) \rangle.
\]

Write \( u := (1, \ldots, 1) \). The equivalence class \( [u] \in H_1 \) is non-zero, but \( 2[u] = 0 \). **Exercise.** The map

\[
\mathbb{Z}^{k-1} \oplus \mathbb{Z} / 2 \to H_1 M_k, \quad (n_1, \ldots, n_{k-1}, m) \mapsto n_1 e_1 + \cdots + n_{k-1} e_{k-1} + mu
\]

is an isomorphism. The point is that there is always a unique expression \( (a_1, \ldots, a_k) = (n_1, \ldots, n_{k-1}, 0) + mu \).
The degree of each of the maps \( f = \psi_\beta \circ \phi_\alpha : S^1 \to S^1 \) can be computed by looking at a point \( q \in S^1 \) which is not the base point. The preimage \( f^{-1}(q) \) will be finite, and the local degree at each \( p \in f^{-1}(q) \) must be \( \pm 1 \), because each \( p \) has a neighborhood \( U \) such that \( f|_U : U \to f(U) \) is a homeomorphism.

**Sphere.** The “usual” CW-structure on \( S^n \) is given by \( S^n \approx D^n/S^{n-1} \). In this case, \( C_n^{\text{CW}} = \mathbb{Z} = C_0^{\text{CW}} \), with all other groups 0.

We also have the “hemispherical” CW structure on \( S^n \). Define a CW-structure on \( S^n \) with two cells \( e_{k,+} \) and \( e_{k,-} \) in each dimension \( 0 \leq k \leq n \), corresponding to the two “hemispheres” of \( S^k \subset \mathbb{R}^{k+1} \times 0 \subset \mathbb{R}^{n+1} \). So we have characteristic maps \( \Phi_{k,+} : D^k \to S^n \) and \( \Phi_{k,-} : D^k \to S^n \).

The skeleton are \( X_k = S^k \subset S^n \).

We choose \( \Phi_{k,+} \) to be a homeomorphism \( D^k \approx \{ (x_0, \ldots, x_k, 0, \ldots, 0) \in S^n \mid x_k \geq 0 \} \) between the \( k \)-disk and the “northern hemisphere” of the \( k \)-sphere, and we can do this so that the restriction \( \phi_{k,+} : S^{k-1} \to S^n \) is the “standard” inclusion \( \phi_{k,+}(x_0, \ldots, x_k) = (x_0, \ldots, x_k, 0, \ldots, 0) \). Then we can let \( \Phi_{k,-} = A \circ \Phi_{k,+} \), which means that \( \phi_{k,-} = A \circ \phi_{k,+} \).

(Here \( A : S^n \to S^n \) is the antipodal map.)

The cellular chains are

\[
\mathbb{Z}\{e^n_0, e^n_1\} \to \mathbb{Z}\{e^{n-1}_0, e^{n-1}_1\} \to \cdots \to \mathbb{Z}\{e^1_0, e^1_1\} \to \mathbb{Z}\{e^0_0, e^0_1\}.
\]

Understanding the boundary maps amounts to understanding the orientations of each of the cells. This is (1) difficult, and (2) not actually worth worrying about. What is obvious from the local degree formula is that

\[
e^k_+ \to \pm e^{k-1}_+ \pm e^{k-1}_-,
\]

but the signs are tricky. We’ll pick orientations which give the formulas we want, using the fact that we already know \( H_* S^n \).

Orient \( D^1 \) so \( e^1_+ \to e^0_+ - e^0_- \). Since \( \Phi_{1,-} = A \circ \Phi_{1,+} \), we get \( e^1_- \to e^0_- - e^0_+ \). (Picture.)

Thus \( e^1_+ + e^1_- \to 0 \), and that \( Z_1^{\text{CW}} \) is generated by this element. If \( n = 1 \), we are done. If not, \( H_1 S^n = 0 \). So choose the orientation on the 2-cell (e.g., by replacing \( \Phi_{2,+} \) with precomposition with a reflection of \( D^2 \)) so \( e^2_+ \to e^1_+ + e^1_- \).

The antipodal map \( A \) takes cells to cells. In particular, \( A_*(e_{k,+}) = e_{k,-} \). This amounts to the commutativity of

\[
(D^k, S^{k-1}) \xrightarrow{\Phi_{k,+}} (S^k, S^{k-1}) \xrightarrow{A} (S^k, S^{k-1}) \xrightarrow{\phi_{k,-}} (D^k, S^{k-1})
\]

Because \( A : S^n \to S^n \) carries cells to cells in the CW-structure, it commutes with \( \partial^{\text{CW}} \). So

\[
\partial^{\text{CW}}(e^2_-) = \partial^{\text{CW}}(A_*(e^2_+)) = A_* \partial^{\text{CW}}(e^2_+) = A_*(e^1_+ + e^1_-) = e^1_+ + e^1_-.
\]

Thus \( e^2_+ - e^2_- \to 0 \) is a cycle. Continuing this way, we can choose \( e^3_+ \to e^2_+ - e^2_- \), and thus \( e^3_- \to e^2_- - e^2_+ \).

Thus,

\[
e^k_+ \to e^{k-1}_+ + (-1)^k e^{k-1}_- , \quad e^k_- \to e^{k-1}_- + (-1)^k e^{k-1}_+.
\]

**Remark.** If a group \( G \) acts on a space \( X \), and we give \( X \) a CW structure \( \{ \Phi^k_\alpha \} \) so that \( G \) permutes the characteristic maps of the cells (i.e., for all \( \alpha \) and \( g \in G \), there is a \( \beta \) such that \( \Phi^k_\alpha \circ g = \Phi^k_\beta \)), we say that \( X \) is a \( G \)-CW complex.
Real projective space. Let $\sigma: S^n \to S^n$ be the antipodal map, and let $\pi_n: S^n \to \mathbb{RP}^n$ be the canonical quotient map, so that $\pi_n \sigma = \pi_n$.

A CW-structure on the $\mathbb{RP}^n$s can be described as follows: $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching a single $n$-dimensional cell $e^n$, by means of an attaching map $\phi_n: S^n \to \mathbb{RP}^{n-1}$.

Claim. We can choose $\phi_n = \pi_{n-1}$.

This gives a CW-structure on $\mathbb{RP}^n$, with one cell $e^k$ in each dimension, with characteristic map $\Phi_k = \pi \circ \Phi_{k+} = \pi \circ \Phi_{k-}: D^k \to \mathbb{RP}^n$. The attaching map is $\phi_k = \pi \circ \phi_{k+} = \pi \circ \phi_{k-}$, which can be identified with the quotient map $\pi: S^k \to \mathbb{RP}^k$. We note the fact that

$$\phi_k = \phi_k \circ \sigma.$$

To compute the boundary map $C_k^{CW} \to C_{k-1}^{CW}$, we need to compute the degree of $f = f_k = \psi_{k-1} \circ \phi_k: S^{k-1} \to \mathbb{RP}^{k-1} \to S^{k-1}$.

Note that over $S^{k-1} - \{\ast\} \approx \mathbb{RP}^k - \mathbb{RP}^{k-1}$, the map $f$ is a 2-fold cover. Let $q \in S^{k-1} = \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ be a point which is not the basepoint. There are two points $\{p, -p\}$ in the preimage of $q$. Since $f$ is a local homeomorphism away from the basepoint of $S^{k-1}$, we have that $\deg f|_p$ and $\deg f|-p$ are $\pm 1$. Write $\deg f|_p = d \in \{\pm 1\}$. Since $f \circ \sigma = \psi_{k-1} \circ \phi_k \circ \sigma = \psi_{k-1} \circ \phi_k = f$, we have that

$$\deg f|-p = \deg(f \circ \sigma)|-p = (\deg f|_p)(\deg \sigma|-p) = (\deg f|_p)(\deg \sigma) = (-1)^k(\deg f|_p).$$

Here I’ve used the fact

Lemma. If $g: S^n \to S^n$, $f: S^n \to S^n$, and $p, q \in S^n$ such that $g(p) = q$, then $\deg(f \circ g)|p = (\deg g|_p)(\deg f|_q)$ whenever these local degrees are all defined.

Also, since the antipodal map is a homeomorphism, $\deg \sigma = \deg \sigma|p$ for any point $p$.

Thus, for $f = \psi_{k-1} \circ \phi_k$, we have

$$\deg f = (\deg f|_p + \deg(f|-p) = (1 + (-1)^k)(\deg f|_p) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \pm 2 & \text{if } k \text{ is even.} \end{cases}$$

Thus the chain complex $C_k^{CW}(\mathbb{RP}^n)$ has the form

$0 \leftarrow \mathbb{Z}\{e^0\} \leftarrow \mathbb{Z}\{e^1\} \leftarrow \mathbb{Z}\{e^2\} \leftarrow \mathbb{Z}\{e^3\} \leftarrow \cdots \leftarrow \mathbb{Z}\{e^n\} \leftarrow 0$

Read off $H_k(\mathbb{RP}^n)$ from this (for odd and even $n$).

Another way to do this is by considering the CW-structure on the sphere. The complex $C_k^{CW}(S^n)$ has the form

$0 \leftarrow \mathbb{Z}\{e^0\} \leftarrow \mathbb{Z}\{e^1\} \leftarrow \mathbb{Z}\{e^2\} \leftarrow \cdots \leftarrow \mathbb{Z}\{e^n\} \leftarrow 0$

Before we begin, note that we already know the homology of the sphere, so this complex is exact except in dimensions 0 and $n$.

The antipodal map $\sigma: S^n \to S^n$ permutes cells, so that $\sigma(e^k_+) = e^k_-$. I claim that $\sigma$ induces an involution of the chain complex, so that $\sigma \partial = \partial \sigma$. This is because $\sigma$ is a cellular map.

We compute $\partial(e^1_+) = e^0_+ - e^0_-$ directly (possibly reversing the orientation on $e^1_+$ to make this work), and thus $\partial(e^1_-) = e^0_- - e^0_+$. Thus $Z_1^{CW} = \mathbb{Z}\{e^0_+ + e^1_+\}$. Assuming $n \geq 2$, we know that $H_1(S^n) = 0$, so we must have $\partial(e^2_+) = \pm (e^1_+ + e^1_-)$. Fix the orientation of the 2-cell, so $\partial(e^2_+) = e^1_+ + e^1_-$, so $\partial(e^2_-) = e^1_- + e^1_+$. Continuing this way, we have

$$\partial(e^k_+) = e^{k-1}_+ + (-1)^ke^{k-1}_- = (-1)^k\partial(e^k_-).$$

The map $\pi: S^n \to \mathbb{RP}^n$ is a cellular map, and it is clear that the induced map $C_k^{CW}(S^n) \to C_k^{CW}(\mathbb{RP}^n)$ sends $e^k_+ \mapsto e^k$ and $e^k_- \mapsto e^k$. From this we deduce that $\partial(e^k) = (1 + (-1)^k)e^{k-1}$. 

Complex projective space. We can describe \( \mathbb{CP}^n \) as the quotient \((\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* \), i.e., the quotient of the action \( \lambda \cdot (z_1, \ldots, z_{n+1}) = (\lambda z_1, \ldots, \lambda z_{n+1}) \). There is a sequence of inclusions \( \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \), and the complement is homeomorphic to \( \mathbb{C}^n \approx \mathbb{R}^{2n} \), by \((z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 1) \). This suggests that \( \mathbb{CP}^n \) is obtained from \( \mathbb{CP}^{n-1} \) by attaching a 2\( n \)-cell, but to prove this we need a characteristic map.

We can define characteristic maps \( \Phi^k : D^{2k} \to \mathbb{CP}^k \) by
\[
\Phi^k(w) = [w, \sqrt{1 - |w|^2}], \quad w \in \mathbb{C}^k, \quad |w| \leq 1.
\]
This leads immediately to the calculation of \( H_* \mathbb{CP}^n \), since \( \mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n} \).

Cubes. The interval \( I \) has a CW-structure with two 0-cells: \( a_0, a_1 \), and one 1-cell: \( b \). The boundary operator can be written
\[
\partial^{\text{CW}}(b) = a_1 - a_0.
\]

The \( n \)-cube \( I \) has a CW-structure with cells that I will denote \( e_1 \times \cdots \times e_n \), with \( e_i \in \{a_0, a_1, b\} \). We can do this by choosing a standard homeomorphism \( I^k \approx D^k \) for all \( k \), and then using the “obvious inclusions” \( I^k \to I^n \) to get the characteristic maps. (Picture.)

It is clear from the local degree formula that \( \partial^{\text{CW}} \) of a \( k \)-cell has every \((k-1)\)-cell with coefficient \( \pm 1 \). Exercise: in \( \partial^{\text{CW}}(b \times \cdots \times b) \) opposite faces must appear with opposite signs. (This is because local degrees on opposite faces can be compared using a reflection of \( I^n \) along a single hyperplane, which is itself a degree -1 map of \((I^n, \partial I^n)\).)

We can choose this structure (for all \( n \)) so that for each \( k = 1, \ldots, n \) and \( i \in \{0, 1\} \), the map \( I^{n-1} \to I^n \) by \((x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{k-1}, i, x_k, \ldots, x_{n-1})\) preserves all the characteristic maps. When you do this, you can deduce all the formulas for the boundary operators. These have the inductive form:
\[
\partial^{\text{CW}}(a_i \times x) = a_i \times \partial^{\text{CW}}(x), \quad \partial^{\text{CW}}(b \times x) = a_1 \times x - a_0 \times x - b \times \partial^{\text{CW}}(x),
\]
where \( x \) is a cell in \( I^{n-1} \). (Example of \( n = 2 \).)

The cellular chain complex for \( I^n \) has the form
\[
\mathbb{Z} \to \mathbb{Z}^{2n} \to \mathbb{Z}^{2^2(\binom{n}{2})} \to \mathbb{Z}^{2^3(\binom{n}{3})} \to \cdots \to \mathbb{Z}^{2^{n-1}(\binom{n}{n-1})} \to \mathbb{Z}^2.
\]
Relate this to \( \sum_k (-1)^{n-k}2^k \binom{n}{k} = (2-1)^n = 1 \).

Tori. Let \( X = S^1 \times \cdots S^1 \) (\( m \)-times). This has a CW-structure coming from the product of the simple \( S^1 \)-structure on \( S^1 \). So the open cells are \( e^{k_1} \times \cdots \times e^{k_m} \) with \( k_i \in \{0, 1\} \). The cellular chains look like
\[
\mathbb{Z} \to \mathbb{Z}^{m-1} \to \mathbb{Z}^{m-2} \to \cdots \to \mathbb{Z}^{1} \to \mathbb{Z}^m \to \mathbb{Z}.
\]

The claim is that all the boundary maps are 0. You can prove this inductively for dimensions below the top, by using cellular inclusions \((S^1)^{m-1} \to (S^1)^m \). We still need to compute \( C^m_{\text{CW}}(X) \to C^{m-1}_{\text{CW}}(X) \). Do this using local degrees; to compare local degrees on opposite sides of a cube, use a reflection.

Another proof: let \( X \) be a space, and \( Y = X \times S^1 \). The inclusion \( X = X \times \{\ast\} \to Y \) has a retraction \( Y \to X \), so the exact sequence of a pair splits, giving
\[
H_q(X \times S^1) \approx H_q(X) \oplus H_q(X \times \{\ast\}).
\]
Note that \((X \times S^1, X \times \{\ast\})\) is a good pair (because \((S^1, \ast)\) is), and that \( Y/X \approx X \times I/X \times \partial I \).
Thus we can compute \( H_*(X,Y) \approx H_*(X \times I, X \times \partial I) \) using a long exact sequence; since \( X \times \partial I \to X \times I \) admits a section up to homotopy, this gives (using the LES of the pair) an isomorphism
\[
H_q(X \times I, X \times \partial I) \approx H_{q-1}(X).
\]
Proposition. \[ H_q(X \times S^1) \approx H_q(X) \oplus H_{q-1}(X) \]
for any space \(X\).

**Lens space.** Let \(k, n\) be relatively prime. The lens space \(L_{k/n}\) is defined by \(D^3/\sim\), where for each point \((x, y, z) \in S^2\) with \(z \geq 0\), we make the identification 
\[ (x, y, z) \sim (R_{k/n}(x, y), -z), \]
where \(R_{k/n}\) represents rotation of the \(xy\)-plane through a turn of angle \(2\pi(k/n)\).

There is a CW-structure, with one 0, 1, 2, and 3 cell, leading to the cellular chain complex
\[ \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}. \]
An equivalent description of \(L_{k/n}\) is \(S^3/G\), where \(S^3 \subset \mathbb{C}^2\) is the unit sphere, and \(G = \langle \sigma \mid \sigma^n \rangle\) acting by
\[ \sigma(z, w) = (\zeta z, \zeta^k w), \quad \zeta = e^{2\pi i/n}. \]
The action is free, so the quotient map \(S^3 \rightarrow S^3/G\) is an \(n\)-fold covering map, and \(S^3/G\) is a compact 3-manifold. It is orientable, since \(\sigma \in G\) acts on \(S^3\) by a map of degree 1.

To see this, consider the subspaces \(B_j = \{(z, \zeta^j(\sqrt{1-|z|^2}) \mid |z| \leq 1\}\} \subset S^3\). Each \(B_j\) is homeomorphic to a 2-disk, with boundary the circle \(C = \{(z, 0) \mid |z| = 1\}\). The group acts so that \(\sigma(B_j) = B_{j+k}\), and \(\sigma\) carries \(C\) to itself by a \(1/n\)-rotation.

**Euler characteristic.** Recall that if \(G\) is a finitely generated abelian group, then \(G \approx \mathbb{Z}^{\text{rank}} \oplus F\) where \(r \geq 0\) and \(F\) is a finite abelian group. The integer \(r\) is called the rank of \(G\), rank and we write \(r = \text{rank} G\).

**Proposition.** If \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) is a short exact sequence of finitely generated abelian groups, then \(\text{rank } B = \text{rank } A + \text{rank } C\).

Let \(C_\bullet\) be a graded abelian group, i.e., a collection \(\{C_q\}_{q \in \mathbb{Z}}\) of abelian groups indexed by an integer \(q\). Suppose that
- \(C_q = 0\) for all but finitely many \(q\), and
- each \(C_q\) is finitely generated.

Then **Euler characteristic** of \(C_\bullet\) is
\[ \chi(C_\bullet) = \sum_k (-1)^k \text{rank } C_k. \]
If \(X\) is a space such that its collection \(H_\bullet(X)\) of homology groups has these properties, then the Euler characteristic of \(X\) is defined to be
\[ \chi(X) := \chi(H_\bullet(X)) = \sum_k (-1)^k \text{rank } H_k(X). \]
Note that Euler characteristic, if defined, is an invariant of the homotopy type of a space: homotopy equivalent spaces have equal Euler characteristics.

**Examples.**
- \(\chi(S^n) = 1 + (-1)^n\). So \(\chi(S^{2n}) = 2\) and \(\chi(S^{2n+1}) = 0\).
- \(\chi(M_g) = 1 - 2g + 1 = 2 - 2g\).
- \(\chi(N_k) = 1 - (k - 1) = 2 - k\).

**Proposition.** If \(X\) has a well-defined Euler characteristic, and \(Y = X \cup_{S^{n-1}} D^n\) is obtained from \(Y\) by attaching an \(n\)-dimensional cell, then \(\chi(Y) = \chi(X) + (-1)^n\).
Proof. We have $H_q(Y, X) \approx H_q(D^n, S^{n-1})$, which is non-trivial only if $q = n$. From the LES of the pair we have $H_q X \approx H_q Y$ unless $q = n - 1, n$, in which case we have an exact sequence

$$0 \to H_{n-1}X \to H_nY \xrightarrow{\alpha} \mathbb{Z} \to H_{n-1}X \to H_{n-1}Y \to 0.$$  

We can think of this as several short-exact sequences “spliced” together:

$$0 \to H_nX \to H_nY \to A \to 0, \quad 0 \to A \to \mathbb{Z} \to B \to 0, \quad 0 \to B \to H_{n-1}X \to H_{n-1}Y \to 0.$$ 

Either

- rank $A = \text{rank } \mathbb{Z} = 1$ and rank $B = 0$, so that
  $$\text{rank } H_nY = 1 + \text{rank } H_nX, \quad \text{rank } H_{n-1}Y = \text{rank } H_{n-1}X,$$

  or

- rank $B = \text{rank } \mathbb{Z} = 1$ and rank $A = 0$, so that
  $$\text{rank } H_nY = \text{rank } H_nX, \quad \text{rank } H_{n-1}Y = \text{rank } H_{n-1}X - 1.$$ 

Either way, $\chi(Y) - \chi(X) = (-1)^n$. \hfill $\square$

Lemma. If $C_\bullet$ is a chain complex such that its Euler characteristic can be defined, then $\chi(C_\bullet) = \chi(H_\ast C_\bullet)$.

Proof. Remember that there are short exact sequences

$$0 \to Z_k \to C_k \to B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to H_k \to 0$$

where $H_k = H_k(C_\bullet)$. Then

$$\text{rank}(C_k) = \text{rank}(Z_k) + \text{rank}(B_{k-1}) = \text{rank}(H_k) + \text{rank}(B_k) + \text{rank}(B_{k-1}).$$

Taking alternating sums of $\text{rank}(C_k)$ cancels the terms $\text{rank}(B_k)$, so we get that

$$\sum (-1)^k \text{rank}(C_k) = \sum (-1)^k \text{rank}(H_k).$$ \hfill $\square$

This means we get an immediate formula for the Euler characteristic of a finite CW-complex, i.e., one with finitely many cells.

Proposition. Let $X$ be a finite CW-complex, and let $c_k$ be the number of $k$-cells in the CW-structure. Then $\sum_k (-1)^k c_k = \chi(X)$. In particular, the number $\sum_k (-1)^k c_k$ does not depend on the choice of finite CW-structure.

This has a famous consequence: any simply connected polyhedron satisfies $V - E + F = \chi(S^2) = 2$, where $V$, $E$, $F$ are the number of vertices, edges, and faces respectively.

Example. Suppose $p: Y \to X$ is an $m$-fold covering map, and $X$ a finite CW-complex. Then we know that $Y$ is also a finite CW-complex, so that above each cell of $X$ there are exactly $m$ cells in $Y$. Thus $c_k(Y) = mc_k(X)$ (numbers of $k$-cells), and therefore $\chi(Y) = m\chi(X)$.

For instance, suppose $X = M_g$ is a closed orientable surface, and $Y \to X$ an $m$-fold covering map with $Y$ connected. Then $Y$ is also a closed surface, and is also orientable. So $Y = M_{g'}$ where

$$2 - 2g' = \chi(Y) = m\chi(X) = m(2 - 2g) \implies g' = 1 + m(g - 1).$$

For instance, a connected finite cover of a genus 1 surface (torus) is also a torus. A connected finite cover of a genus 3 surface is always a surface of odd genus.
Homology with coefficients. Given an abelian group $G$, the singular chains with coefficients in $A$ are
\[ C_q(X; G) \overset{\text{def}}{=} C_q(X) \otimes G \approx \bigoplus_{\sigma : \Delta^n \to X} G, \]
so an element of this group is a formal sum
\[ \sum_{\sigma : \Delta^n \to X} g_{\sigma} \sigma, \]
where $g_{\sigma} \in G$ and $g_{\sigma} = 0$ for all but finitely many $\sigma$.

The boundary map is $\partial^G \overset{\text{def}}{=} \partial \otimes \text{id}$, which has the formula
\[ \partial^G(\sum_{\sigma} g_{\sigma} \sigma) = \sum_{\sigma} \sum_{k=0}^{n} (-1)^k g_{\sigma} (\sigma[v_0, \ldots, \hat{v}_k, \ldots, v_n]) \]
for $g \in G$ and $\sigma : \Delta^n \to X$.

Note that $C_*(X; \mathbb{Z}) = C_*(X)$. We get a relative chain complex $C_*(X, A; G) = C_*(X/G)/C_*(A; G)$ in the same way.

Singlar homology with coefficients in $G$ is defined by
\[ H_q(X, A; G) \overset{\text{def}}{=} H_* C_*(X, A; G). \]
This satisfies all the Eilenberg-Steenrod axioms, except that the dimension axiom is modified in the obvious way. We can similarly define $H^k(X; G)$ and $H^k_{\text{CW}}(X; G)$, and these agree with the singular theory, by the same proofs we gave earlier.

Typically one takes $G = \mathbb{Z}/n$ or $G \subseteq \mathbb{Q}$.

**Example.** $H_*(\mathbb{R}^n; \mathbb{Z}/2)$ and $H_*(\mathbb{R}^n; \mathbb{Z}/p)$ for $p$ odd.

**Remark.** If $F$ is a field, then we have a notion of dimension of $F$-vector space. Thus we can consider $\sum_k (-1)^k \dim_F H^k(X; F)$. If $X$ is a finite CW-complex, then this is equal to $\chi(X)$.

The relation between $H_*(-; G)$ and $H_*(-)$ is entirely determined by the “universal coefficient theorem”, which we leave for 526. In short, $H_q(X; G)$ depends on both $H_q(X)$ and $H_{q-1}(X)$. (See exercises §2.2 #40 and #43 in Hatcher for examples when $G = \mathbb{Z}/n$.)

**Homology and the fundamental group.** Let $(X, x_0)$ be a pointed space. Let $\iota_1 : \Delta^1 \to I$ be the evident linear homeomorphism, sending $v_k \mapsto k$ for $k = 0, 1$. Note that $[\iota_1] \in H_1(I, \partial I)$ is a generator.

Let $f : I \to X$ be a loop based at $x_0$. Define $h(f) \in \tilde{H}_1(X) = H_1(X, \{x_0\}) = H_1(X)$ by $h(f) := f_*[\iota_1] = [f\iota_1]$, where $f : (I, \partial I) \to (X, \{x_0\})$ is the induced map of pairs. Using the homotopy axiom, we see that $h(f)$ only depends on the path homotopy type of $f$, so we get a function
\[ h : \pi_1(X, x_0) \to \tilde{H}_1(X). \]
This function $h$ is a group homomorphism. To see this, suppose we have loops $f_0, f_1, f_2$ with $f_1 \sim f_2 * f_0$. Then we can construct a map
\[ g : (\Delta^2, \{e_0, e_1, e_2\}) \to (X, \{x_0\}) \]
such that $g \circ d^2 = f_1 \circ \iota_1$. In $C_1(X)$ we have $\partial(g) = f_0\iota_1 - f_1\iota_1 + f_2\iota_1$, whence $h(f_1) = h(f_2) + h(f_0)$.

The homomorphism $h : \pi_1(X, x_0) \to H_1(X)$ is called the Hurewicz homomorphism.

Recall that if $G$ is a group, then $G^{ab} := G/[G, G]$ is the maximal abelian quotient of $G$. 

Hurewicz homomorphism
maximal abelian quotient

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Hurewicz homomorphism
maximal abelian quotient
Theorem (Hurewicz). If $X$ is path connected, then $h$ surjective, with kernel equal to the commutator subgroup. That is, $h$ induces an isomorphism

$$\pi_1(X, x_0)^{ab} \approx H_1(X)$$

for all path connected $X$.

Idea of proof for a CW-complex. Let $X$ be a connected CW-complex, with a single 0-cell $\{x_0\}$. We know that $\pi_1(X, x_0) \approx \langle a_i \mid r_j \rangle$, where there is a generator $a_i$ for each 1-cell, and a relation $r_j$ for each 2-cell. On the other hand, we have shown that $H_1 X = \text{Cok}[C_2 \to C_1]$; i.e., it is an abelian group generated by 1-cells, with relations coming from 2-cells. This shows that

$$\pi_1(X, x_0)^{ab} \approx H_1(X).$$

If you trace through the definitions, you see that the isomorphism coincides with the map $h$ defined above.

If $X$ has more than one 0-cell, find a maximal tree $T \subseteq X_1$. Then $X \to X/T$ is a homotopy equivalence, and the above argument applies with $X$ replaced by $X/T$.

A useful construction. To prove this we will need a way to “lift” singular chains to simplicial chains in specially constructed Delta complexes.

Lemma. Let $S = \{\sigma_\alpha : \Delta^{n_\alpha} \to X\}_\alpha$ be a set of singular simplices in $X$, such that $S$ is closed under face operators: i.e., for all $j = 0, \ldots, n_\alpha$ we have $\sigma_\alpha \circ d^j \in S$. Then there exists a $\Delta$-complex $K$ and a map $f : K \to X$ such that

1. the collection $\tilde{S} = \{\tilde{\sigma}_\alpha : \Delta^{n_\alpha} \to K\}$ of simplices in the $\Delta$-complex structure on $K$ is in bijective correspondence with $S$, so that
2. $f \circ \tilde{\sigma}_\alpha = \sigma_\alpha$ as maps $\Delta^{n_\alpha} \to X$.

Furthermore, $f$ induces an injective chain map $f_\# : \Delta_*(K) \to C_*(X)$ sending $\tilde{\sigma}_\alpha \mapsto \sigma_\alpha$.

Proof. Define $K := \bigsqcup_\alpha \Delta^{n_\alpha} / \sim$, where for each $\alpha$ and $j = 0, \ldots, n_\alpha$ we identify the $j$th face of the summand $\Delta^{n_\alpha}$ corresponding to $\beta$ with the summand $\Delta^{n_\beta}$ corresponding to $\beta$, where $\sigma_\beta = \sigma_\alpha \circ d^j$. Write $\tilde{\sigma}_\alpha$ for the evident map from the $\alpha$-summand to the quotient. Then $f$ is the unique map such that $f \circ \tilde{\sigma}_\alpha = \sigma_\alpha$, which is defined exactly because of the identifications made in defining $K$.

Corollary. Let $c \in C_n X$ be a singular chain. Then there exists an $n$-dimensional $\Delta$-complex $K$, a continuous map $f : K \to X$, and a simplicial chain $\tilde{c} \in \Delta_n X$ such that

1. $f_\# : \Delta_* K \to C_* X$ is injective, and
2. $f_\#(\tilde{c}) = c$.

In particular, if $c$ is a cycle, so is $\tilde{c}$.

Proof. Write $c = \sum_{i=1}^r n_i \sigma_i \in C_n X$. Apply the previous lemma to the set $S$ obtained as the closure of $\{\sigma_1, \ldots, \sigma_r\}$ under face operations.

Note that if $c$ is a cycle, $f_\# \partial \tilde{c} = \partial f_\# \tilde{c} = \partial c = 0$, so injectivity of $f_\#$ implies $\tilde{c}$ is also a cycle.

A smaller description of $H_q X$ for path connected $X$. Fix our path-connected based space $(X, x_0)$. We note that we can compute $H_1 X$ using only 1-chains built from loops at $x_0$, and 2-chains whose faces are loops at $x_0$.

For any $q \geq 0$, let

$$C_q^X := \{ \sum n_i \sigma_i \in C_1 X \mid \sigma_i : \Delta^q \to X \text{ satisfies } \sigma_i(v_j) = x_0 \text{ for } j = 0, \ldots, q, \}$$
the subgroup of $C_qX$ spanned by singular simplices whose vertices are all sent to the basepoint $x_0$. Note that the boundary operator restricts to these subgroups, so we get a subcomplex $C'_qX \subseteq C^*_X$. Observe $C'_1X$ is a subgroup of the group $Z_1X \subseteq C_1X$ of 1-cycles, and in fact is naturally isomorphic to the free abelian group on the set of loops at $x_0$.

Write $H'_qX := H_q[\mathcal{C}'_qX]$.

**Lemma.** For path connected $(X, x_0)$, the inclusion $C'_*X \subseteq C^*_X$ of chain complexes induces isomorphisms $\phi: H'_qX \sim H_qX$.

**Proof.** $\phi$ is surjective. Given $c \in C_qX$ with $\partial c = 0$, we want to find $c' \in C'_qX$ with $\partial c' = 0$ such that $[c] = [c']$ as elements of $H_qX$. Using the Corollary above, construct $f: K \to X$ with $K$ a $\Delta$-complex, such that $f# : \Delta_0K \to C'_*X$ is injective and $c = f#(\tilde{c})$ for some cycle $\tilde{c} \in \Delta_qK$.

Since the 0-skeleton $K_0$ is discrete and $X$ is path connected, there exists a homotopy of maps $K_0 \to X$ between $f|K_0$ and the constant map $g|K_0 : K_0 \to \{x_0\} \to X$. By the homotopy extension property applied to $K_0 \to K$, there thus exists a map $g : K \to X$ homotopic to $f$ such that $g(K_0) \subseteq \{x_0\}$. Then $c' = g#(\tilde{c})$ is the desired cycle, since by the homotopy axiom $[c'] = g#(\tilde{c}) = f#([\tilde{c}]) = [c]$.

$\phi$ is injective. Suppose $c \in C'_qX$ such that $c = \partial b$ for some $b = \sum_j u_j \sigma_j \in C_{q+1}X$. We want to find $b' \in C'_qX$ such that $c = \partial b'$. Using the Corollary, construct $f : X \to X$ with $K$ a $\Delta$-complex such that $f# : \Delta_0K \to C'_*X$ is injective, and $b = f#(\tilde{b})$ for some chain $\tilde{b} \in \Delta_{q+1}K$.

Write $\tilde{c} := \partial \tilde{b} = \sum m_j \tilde{\sigma}_j \in \Delta_1K$, and assume all $m_j \neq 0$. Then $c = f#(\tilde{c}) = \sum m_j \sigma_j \in C'_qX$. Let $K^c := \bigcup_j \tilde{\sigma}_j(\Delta^q)$, and set $L := K_0 \cup K^c$, a sub-$\Delta$-complex of $K$. Note that $L$ is a disjoint union

$$L = K^c \sqcup S, \quad S = L \setminus K^c,$$

where $S \subseteq K_0$ is a set of discrete points.

Thus since $X$ is path connected, there exists a homotopy rel $K^c$ between $f|L : L \to X$ and a map “$g|L$” which sends $K_0$ into $\{x_0\}$. Using the homotopy extension property applied to $(K, L)$, we get a map $g : K \to X$ homotopic to $f$ such that $g(K_0) = \{x_0\}$ and $g|K^c = f|K^c$, so $g#(\tilde{c}) = c$. Then $b' := g#(\tilde{b}) \in C'_qX$ and satisfies $\partial(b') = c$. \hfill $\square$

**Proof of the Hurewicz theorem.** Fix a path connected $(X, x_0)$, and write $G := \pi_1(X, x_0)$. The construction of the Hurewicz homomorphism actually uses only our “smaller” construction of $H_1$. We have a commutative diagram

$$\begin{array}{ccc}
\{\text{Loops at } x_0\} & \longrightarrow & G \\
\downarrow & & \downarrow \tilde{h} \\
C'_2X & \xrightarrow{\partial} & C'_1X \\
\downarrow & \downarrow \pi & \downarrow \sim \\
C_2X & \xrightarrow{\partial} & Z_1X \\
\downarrow & \downarrow \phi & \downarrow \\
\{\text{Loops at } x_0\} & \longrightarrow & H'_1X \\
\end{array}$$

so that the composite $\{\text{Loops at } x_0\} \xrightarrow{\tau} C'_1X \xrightarrow{\pi} H'_1X$ descends to a homomorphism $h : G \to H'_1X$, which further descends to a homomorphism $\tilde{h} : G^{ab} \to H'_1X$. We want to show $\tilde{h}$ is a bijection.

Since $C'_1X$ is the free abelian group on loops at $x_0$, we can define a homomorphism

$$\tau : C'_1X \to G^{ab}$$
by sending a generator \((\gamma: \Delta^1 \to X)\) to the coset in \(G^{ab} = G/[G, G]\) represented by \([\gamma] \in G\).

We note the following.

1. We have that \(\tau\) is surjective: for instance, every element of \(G^{ab}\) is equal to \(\tau(\gamma)\) for some loop \(\gamma: \Delta^1 \to X\) at \(x_0\).
2. We have \(h \circ \tau = \pi\), tautologically.
3. The composite \(C_2'X \xrightarrow{\partial} C_1'X \xrightarrow{\tau} G^{ab}\) is trivial, since \(\partial \sigma = (\sigma d^0) - (\sigma d^1) + (\sigma d^2)\) gives a relation \([\sigma d^2][\sigma d^0] = [\sigma d^1]\) in \(G\).

To show that \(h\) is surjective, it suffices to use \(h\tau = \pi\) and that \(\tau\) is surjective.

To show that \(h\) is injective, consider a class \(u \in G^{ab}\) such that \(h(u) = 0\). Since \(\tau\) is surjective we have \(u = \tau(c)\) for some \(c \in C_1'X\). Then \(\pi(c) = h\tau(c) = 0\), so \(c = \partial b\) for some \(b \in C_2'X\). But then \(u = \tau(c) = \tau(\partial b) = 0\), as desired, since \(\tau \circ \partial = 0\).

**Acyclic space.** Do there exist spaces \(X\) such that \(\tilde{H}_*X \approx 0\) but \(X\) is not contractible? Yes! In fact, there are CW-complexes with this property. (So this is not a “pathological” phenomenon of weird spaces.)

This is Example 2.38 in Hatcher. Build a CW-complex \(X\) with \(X_1 = S^1 \cup S^1\) (called \(a\) and \(b\)), and \(X = X_1 \cup e^2 \cup e^2\), where the 2-cells are attached by \(a^5b^3\) and \(b^3(ab)^{-2}\).

An easy calculation shows that \(\tilde{H}_*(X) = 0\). It is described as the kernel/cokernel of the matrix \(A = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\) acting of \(\mathbb{Z}^2\). Because \(\det A = -1\), it has an inverse over \(\mathbb{Z}\).

But \(X\) is not contractible, because \(\pi_1(X) = \langle a, b | a^5b^3, b^3(ab)^{-2} \rangle\) is not the trivial group; in fact, it is the binary icosahedral group. It is helpful to think of the relations as \(a^5 = b^3 = (ab)^2\). To see that \(\pi_1(X)\) is non-trivial, construct a map \(\rho: \pi_1(X) \to G\) to the group of symmetries of a regular icosahedron (or dodecahedron if you prefer). Here, \(\rho(a)\) is rotation around the axis through a vertex, and \(\rho(b)\) is rotation around the axis through the center of a triangular face which touches this vertex. Then \(\rho(ab)\) is rotation through the midpoint of an edge which has this vertex at one end. So \(\rho(a)^5 = 1, \rho(b)^3 = 1, \rho(ab)^2 = 1\).

The kernel of \(\rho\) turns out to be of order 2, generated by \(a^5 = b^3 = (ab)^2\), so \(|\pi_1X| = 120\), since \(G \approx A_5\) has order 60. In fact, \(\tilde{G} = \pi_1X\) is the binary icosahedral group, which is a central extension of \(G\) by a group of order 2.

**Exercise.** Let \(\tilde{X} \to X\) be the universal cover of \(X\). Show that \(H_0\tilde{X} \approx \mathbb{Z}, H_2\tilde{X} \approx \mathbb{Z}^\oplus 119\), and \(H_q\tilde{X} \approx 0\) for \(q \neq 0, 2\). (Hint: cellular chains and Hurewicz theorem.)

**Remark.** The above example is closely related to the **Poincaré 3-sphere.** The icosahedral group \(G\) is a subgroup of \(SO(3)\) (the group of rotations of \(\mathbb{R}^3\) around the origin), and so we can form the quotient space \(P := SO(3)/G\), which is a closed 3-dimensional manifold. One can show that \(H_*(P) \cong H_*(S^3)\), but that \(\pi_1 P \cong \tilde{G}\) (the binary icosahedral group). The space \(P \setminus \{p\}\) obtained by removing one point is homotopy equivalent to \(X\).

On the other hand, we have the following results about simply connected CW-complexes.

**Proposition.** If \(X\) is a simply connected CW-complex such that \(\tilde{H}_*X \approx 0\), then \(X\) is contractible.

More generally, if \(f: X \to Y\) is a map between simply connected CW-complexes which induces an isomorphism \(f_*: H_*X \to H_*Y\), then \(f\) is a homotopy equivalence.

**Proof.** Math 526/7. \(\square\)

**The Lefschetz trace.** Given an abelian group \(A\), let \(A_{tors} \subseteq A\) be the subgroup of torsion elements (= elements of finite order), and let \(A_{tors} := A/A_{tors}\). The group \(A_{tors}\) is the **torsion-free quotient** of \(A\). If \(A\) is finitely generated, then \(A_{tors} \approx \mathbb{Z}^\oplus \text{rank} A\).
A homomorphism \( f : A \to B \) between finitely generated abelian groups induces \( f_{it} : A_{it} \to B_{it} \). In this case, we define the \textbf{trace} \( \text{Trace} f \) of \( f \) to be the trace of \( f_{it} \). As this is a map between finitely generated free abelian groups, its trace is just the trace of the integer valued matrix representing it, in terms of chosen bases for the groups. One can show (as for maps of vector spaces), that this trace does not depend on choices of bases.

Suppose \( C_* \) is a graded abelian group, such that \( \bigoplus_q C_q \) is finitely generated, i.e., (i) all but finitely many \( C_q \) are trivial, and (ii) and each \( C_q \) is finitely generated. The \textbf{Lefschetz trace} \( \tau(f) \) of a self-map \( f : C_* \to C_* \), i.e., a collection of maps \( f_q : C_q \to C_q \), is defined by

\[
\tau(f) := \sum_k (-1)^k \text{Trace} [f_k : C_k \to C_k].
\]

Example: if \( f = \text{id} \), then \( \tau(f) = \chi(C_*) \). You can think of the Lefschetz trace as a variant of Euler characteristic defined for self-maps.

Suppose \( f : X \to X \) is a self-map of a space \( X \). If the homology \( H_* X = \bigoplus_k H_k X \) is a finitely generated group, the \textbf{Lefschetz number} \( \tau(f) \in \mathbb{Z} \) is defined to be

\[
\tau(f) := \tau(f_* : H_* X \to H_* X) \sum (-1)^k \text{Trace} (f_k : H_k(X) \to H_k(X)).
\]

Note that \( \tau(f) \) only depends on the homotopy class of \( f \), and that \( \tau(\text{id}_X) = \chi(X) \).

**Lefschetz fixed point theorem.** I state the Lefschetz fixed point theorem here.

**Theorem** (Lefschetz). Let \( X \) be a retract of a finite simplicial complex. If \( f : X \to X \) is a self-map which has no fixed point, then \( \tau(f) = 0 \).

We can restate this as: if \( \tau(f) \neq 0 \), then \( f \) has a fixed point.

I’ll describe the hypothesis “retract of a finite simplicial complex” below. It implies that \( X \) is compact, and for the time being, you should think of it as meaning “nice compact space”. First I’ll describe some applications.

**Example** (Brouwer fixed point theorem). The theorem applies to \( X = D^n \). Any \( f : D^n \to D^n \) induces the identity map on homology, which is concentrated in dimension 0, so \( \tau(f) = \chi(D^n) = 1 \). Thus by Lefschetz every \( f \) has a fixed point.

Note that the theorem does not apply to \( X = \mathbb{R}^n, n \geq 1 \), for which we also have \( \tau(f) = 1 \). In this case there are fixed-point free self-maps, e.g., translations.

**Example** (Spheres). The theorem applies to \( X = S^n \). Then \( \tau(f) = 1 + (-1)^n \deg(f) \), and so we recover the fact that if \( f : S^n \to S^n \) is fixed point free, then \( \deg(f) = (-1)^{n+1} = \deg A \).

**Example** (Real projective space). If \( X = \mathbb{R}P^{2n} \), then for any self map \( f \) we have \( \tau(f) = 1 \). So any self map of an even dimensional real projective space has a fixed point.

**Example** (Nowhere-vanishing vector fields). Let \( M \) be a compact smooth manifold, and let \( v \) be a smooth tangent vector field of \( M \). You can “flow” along the vector field, to obtain a 1-parameter family \( f_t : M \times \mathbb{R} \to M \) of self-maps of \( M \), i.e., so that \( \left[ d(f_t(p))/dt \right]_{t=t_0} = v(f_{t_0}(p)) \). There is \( \epsilon > 0 \) such that for \( 0 < t < \epsilon \), the fixed points of \( f_t \) are precisely the points \( p \) where \( v_p = 0 \). The flow \( t \mapsto f_t \) provides a homotopy \( f_t \sim f_0 = \text{id}_M \), so we have that \( \tau(f) = \chi(M) \). An a consequence, if \( v \) is a nowhere vanishing vector-field, then \( \chi(M) = 0 \). In other words, if \( \chi(M) \neq 0 \), every vector field of \( M \) vanishes somewhere.

**Simplicial complexes.** Recall the definition of \textit{simplicial complex}: a \( \Delta \)-complex \( X \) such that every simplex in the \( \Delta \)-complex structure is determined by its set of vertices. That is, given \( \sigma_k : \Delta^k \to X \) with \( k = 1, 2 \) elements of the \( \Delta \)-complex structure, if \( \{ \sigma_1(v_j) \mid j = 0, \ldots, n_1 \} = \{ \sigma_2(v_j) \mid j = 0, \ldots, n_2 \} \), then \( n_1 = n_2 \) and \( \sigma_1 = \sigma_2 \). Note that this implies that each such \( \sigma \)
is an injective map (since if $\sigma(v_i) = \sigma(v_j)$ for some $i \neq j$, then $\sigma$ and $\sigma d^k$ have the same sets of vertices).

**Remark.** A $\Delta$-complex $X$ is a simplicial complex if and only if it admits an embedding $f: X \to \mathbb{R}^N$ into a (possibly infinite dimensional) Euclidean space such that $f \sigma: \Delta^n \to \mathbb{R}^N$ is affine linear for every $\sigma: \Delta^n \to X$ in the $\Delta$-complex structure.

For instance, if $X$ is a simplicial complex with vertex set $V$, define $f: X \to \mathbb{R}^V$ so that each vertex $v$ of $X$ is sent to the basis vector $e_v$, and extend to higher dimensional simplices by linearity. The converse is straightforward.

**Retracts.** We say that $X$ is a retract of $Y$ if there exists $i: X \to Y$, $r: Y \to X$ such that $ri = id_X$.

This implies that $i$ is an embedding, i.e., $X$ is homeomorphic to the subspace $i(X) \subseteq Y$, so we often talk about a subspace $X \subseteq Y$ being a retract. Note that not every embedding is a retract, e.g., $i: S^{n-1} \to D^n$.

**Example** (Euclidean neighborhood retract). A space $X$ is a Euclidean neighborhood retract (ENR) if there exists an embedding $i: X \to \mathbb{R}^n$ (for $n$ finite) and a neighborhood $U$ of $X$ in $\mathbb{R}^n$ such that $i(\overline{X})$ is a retract of $U$. See Hatcher, Appendix A, for more on these.

Every compact manifold (Hatcher A.9) and every finite CW-complex (Hatcher A.10) is an ENR. Furthermore, a space is a compact ENR iff it is a retract of a finite simplicial complex (Hatcher A.8).

If $X$ is a retract of $Y$, with maps $i: X \to Y$ and $r: Y \to X$ such that $ri = id_X$, we have a direct sum decomposition

$$H_qY = \text{Im } i_* \oplus \text{Ker } r_*, \quad \text{Im } i_* \approx H_qX.$$  

If $f: X \to Y$ is a self-map, we can define $g := i fr: Y \to Y$. Then in terms of the above decomposition $g_*: H_qY \to H_qY$ is given by the matrix $egin{pmatrix} f_* & 0 \\ 0 & 0 \end{pmatrix}$. (The proof is straightforward.) In particular, $\tau(g) = \tau(f)$ if $\tau(g)$ is defined. Furthermore, the fixed points of $g$ are precisely the fixed points of $f$ (since $ifr(y) = y$ implies $y = i(x)$ for a unique $x \in X$, and that $f(x) = x$).

Thus, if the conclusion of the Lefschetz fixed point theorem holds for any self-map of $Y$, then it holds for any self-map of $X$. So we only need to prove the theorem for finite simplicial complexes.

**A weak Lefschetz theorem for cellular maps.**

**Lemma.** If $C_*$ is a chain complex of finitely generated abelian groups with $\bigoplus C_k$ finitely generated, and $f: C_* \to C_*$ is a chain map, then $\tau(f) = \tau(H_* f)$.

In particular, if $f: X \to X$ is a cellular self-map of a finite CW-complex, then $\tau(f) = \tau(f#: C_*^{CW} X \to C_*^{CW} X)$.

**Proof.** Verify that given a self map of a short exact sequence $0 \to A \to B \to C \to 0$ of finitely generated abelian groups, that $\text{Trace}(f_B) = \text{Trace}(f_A) + \text{Trace}(f_C)$. The claim follows by an argument similar to the one we gave for Euler characteristic. \qed

Here is a very restricted case of the Lefschetz theorem. Remember that for a CW-complex $X$, we have a decomposition $X = \bigsqcup e^k_\alpha$ as a set (not as a space), where $e^k_\alpha = \Phi_\alpha(\text{Int } D^n) \subseteq X$.

**Proposition.** Let $f: X \to X$ be a cellular map of a finite CW-complex. Suppose that for each cell $e^k_\alpha \subseteq X$, we have that $f(e^k_\alpha) \cap e^k_\alpha = \emptyset$. Then $\tau(f) = 0$. 

Proof. Since $f$ is cellular it induces a map on cellular chains $f_\#: C^\bullet_{CW}(X) \to C^\bullet_{CW}(X)$. Write

$$f_\#(e_\alpha) = \sum f_{\alpha\beta} e_\beta, \quad f_{\alpha\beta} \in \mathbb{Z}.$$ 

There is a degree formula for the coefficients, whose proof we leave for the reader:

$$f_{\alpha\beta} = \deg[S^n \approx D^n/S^{n-1} \xrightarrow{\Phi_\alpha} X_n/X_{n-1} \xrightarrow{f} X_n/X_{n-1} \xrightarrow{\psi_\beta} S^n].$$

In particular, $f_{\alpha\alpha} = 0$, since by hypothesis $f(\Phi_\alpha(D^n)) \subseteq X_n \setminus e^n_\alpha$, so $f_{\alpha\alpha}$ is the degree of a constant map.

Note: the hypothesis on $f$ implies it has no fixed points, but is rather stronger than that.

Our proof of Lefschetz proceeds as follows: given a finite simplicial complex $X = K$ and a map $f : X \to X$ with no fixed points, we will produce:

- a new simplicial complex structure $K'$ on $X$,
- a map $g : X \to X$ which is homotopic to $f$, and which
- is cellular
- satisfies $g(e_\alpha) \cap e_\alpha = \emptyset$ for the interiors $e_\alpha$ of all simplices in $K'$.

Then $\tau(f) = \tau(g) = 0$ by the above discussion.

**Simplicial approximation.** Now suppose $X = K$ is a finite simplicial complex, and choose an embedding $K \to \mathbb{R}^N$ into a finite dimensional Euclidean space, so that the restriction to each simplex is an affine linear map.

The subdivision $sd K$ is a new simplicial complex structure on the same space, where the simplices of $sd K$ are elements of the barycentric subdivisions of the simplices of $K$. We can iterate subdivision as many times as we like, obtaining $sd^m K$. Because there are finitely many simplices, then for any $\epsilon > 0$ we can choose $m \geq 0$ such that the diameter of any simplex in $sd^m K$ is less than $\epsilon$.

A simplicial map $f : K \to L$ between simplicial complexes is a continuous map which takes each simplex of $K$ into a simplex of $L$ by a linear map sending vertices to vertices. Note that a simplicial map is automatically a cellular map. Also, note that a simplicial map is determined by its values on vertices. (Unlike in previous discussions involving $\Delta$-complexes, we do not require that $f$ preserve any ordering of vertices.)

The following is a simplicial approximation theorem.

**Proposition.** Let $f : K \to L$ be a continuous map between finite simplicial complexes; suppose $L$ is linearly embedded in Euclidean space, with maximal simplex diameter less than $\epsilon > 0$. Then there exists $m \geq 0$ and a simplicial map $g : sd^m K \to L$ such that $f$ and $g$ are homotopic, and $|g(x) - f(x)| < \epsilon$ for all $x \in K$.

**Idea of proof.** Here is an idea of how this is proved. If $K$ is a simplicial complex, the star $St(\sigma)$ of a simplex $\sigma$ is the union of all closed simplices in $K$ which contain $\sigma$; the open star $st(\sigma)$ is the union of the interiors of the simplicies of $St(\sigma)$. (Picture.)

Fact (Hatcher, Lemma 2C.2): an intersection $\bigcap st(v_i)$ of stars of a collection of vertices is equal to $st(\sigma)$ if the $v_i$’s are the vertices of some simplex $\sigma$, or is empty if they aren’t.

The idea of simplicial approximation is that you define $g$ to be a simplicial map which sends each vertex $v$ of $sd^m K$ to a vertex $w$ of $L$ such that $f(St(v)) \subseteq st(w)$. Thus, $m$ needs to be chosen sufficiently large so this happens, using the Lesbegue number lemma. To show that you have a simplicial map, you need to show that if $(v_0, \ldots, v_q)$ are the vertices of a $q$-simplex $\sigma$ in $sd^m K$, then $(g(v_0), \ldots, g(v_q))$ are the vertices of a simplex in $L$ (possibly of dimension less than $q$). The interior of $\sigma$ is contained in $st(\sigma) = \bigcap st(v_i)$, and thus $f(\sigma)$ is contained in $\bigcap st(g(v_i))$, which is therefore non-empty! 

\[\square\]
Given this, the Lefschetz theorem follows. Let $K$ be a finite simplicial complex linearly embedded in Euclidean space, and consider a continuous map $f: K \to K$ with no fixed point. Since $K$ is compact, there exists $\epsilon > 0$ such that $|f(x) - x| > 2\epsilon$ for all $x \in K$. Choose $m'$ such that $\text{sd}^{m'} K$ has maximal simplex diameter less than $\epsilon$, then choose $m \geq m'$ and a simplicial approximation $g: \text{sd}^{m} K \to \text{sd}^{m'} K$ of $f$, which by construction has the property that $|g(x) - f(x)| < \epsilon$ for all $x$, and thus $|g(x) - x| > \epsilon$. The map $g$ is cellular, viewed as a map $\text{sd}^{m} K \to \text{sd}^{m} K$. Because the maximal simplex diameter of $\text{sd}^{m} K$ is less that $\epsilon$, it follows that $g(\sigma) \cap \sigma = \emptyset$ for every boundaryless simplex $\sigma$ in $\text{sd}^{m} K$.

Thus $\tau(g) = 0$, and so $\tau(f) = 0$ since $f$ and $g$ are homotopic.