(1) Let $0 \to K \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} L \to 0$ be an exact sequence of chain complexes in an abelian category, and suppose that $H_*A \approx 0 \approx H_*B$. Construct isomorphisms $H_nK \approx H_{n+2}L$.

**Solution.** Use the two applications of the long exact sequence in homology. Define $M := \text{Im } g$. Because the sequence is exact, we obtain two exact sequences

$$0 \to K \xrightarrow{f} A \xrightarrow{g} M \to 0, \quad 0 \to M \xrightarrow{d} B \xrightarrow{h} L \to 0$$

with $g = jq$ being the epi/mono factorization of $g$. Each of these sequences provides connecting homomorphisms

$$H_{n+2}L \xrightarrow{\partial_2} H_{n+1}M \xrightarrow{\partial_1} H_nK.$$

Because $H_*A \approx 0$, the long exact sequence in homology of the first short exact sequence gives that $\partial_1$ is an iso; likewise, $H_*B \approx 0$ gives that $\partial_2$ is an iso.

(2) Fix an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of chain complexes in $\text{Ab}$. In addition, suppose that $B_n = C_n \oplus A_n$, $f(a) = (0, a)$, $g(c,a) = c$.

We say that such an exact sequence is “degreewise split-exact”.

(a) Show that the boundary map in $B$ has the form

$$d_B(c,a) = (d_C(c), d_A(a) + h(c))$$

for a collection of maps $h: C_n \to A_{n-1}$, $n \in \mathbb{Z}$.

**Solution.** This is just an application of the identities $d_Bf = fd_A$ and $gd_B = d_Cg$. In fact, consider $H: B_n \to B_{n-1}$ defined by $H(c,a) = (H_1(c,a), H_2(c,a)) := d_B(c,a) - (d_C(c), d_A(a))$. Then

$$Hf(a) = d_Bf(a) - (0, d_A(a)) = fd_B(a) - (0, d_A(a)) = (0, d_B(a)) - (0, d_A(a)) = 0$$

and

$$gH(c,a) = gd_B(a,c) - d_C(c) = d_Cg(a,c) - d_C(c) = d_C(c) - d_C(c) = 0.$$

Thus, $H_1(c,a) = 0$ and $H_2(c,a) = H_2(c,0)$, so $H(c,a) = (0, h(c))$ for some function $h: C_n \to A_{n-1}$.

(b) Show that $d_Ah + hd_C = 0$, and show that $h$ defines a chain map $C[1] \to A$.

**Solution.** The identity follows from $d_Bd_B = 0$. Compute

$$0 = d_Bd_B(c,a) = d_B(d_C(c), d_A(a) + h(c)) = (d_Cd_C(c), d_Ad_A(a) + d_Ah(c) + hd_C(c)) = (0, (d_Ah + hd_C)(c)).$$

Note that $d_{C[1]} = -d_C$, so the identity asserts that $d_Ah = hd_{C[1]}$, i.e., $h: C[1] \to A$ is a chain map.

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(c) Show that the connecting map \( \delta: H_n C \to H_{n-1} A \) in the long exact sequence in homology for \( 0 \to A \to B \to C \to 0 \) coincides with \( H_n C \approx H_{n-1} C[1] \frac{H_{n-1} A}{A} \).

**Solution.** The connecting map is given by: \( \delta([c]) = [a] \) iff there exists \( b \in B_n \) such that \( f(a) = d_B(b) \) and \( g(b) = c \). Using our formula for \( d_B \), we get that \( b = (c, a') \) with \( d_A(a') + h(c) = a \). Thus, \( a \) is homologous to \( h(c) \), so \( \delta([c]) = [h(c)] \).

(d) Give an isomorphism \( B \approx \text{cone}(-h) \) of chain complexes.

**Solution.** According to the definition of the cone of \(-h: C[1] \to A\) given in class, we have

\[
\text{cone}(-h)_n = C[1]_{n-1} \oplus A_n \approx C_n \oplus A_n
\]

with boundary map

\[
d_{\text{cone}(-h)}(x, y) = (-d_{C[1]}(x), d_A(y) - (-h)(x)) = (d_C(x), d_A(y) + h(x)).
\]

(3) Fix a prime \( p \). Let \( C \) be the chain complex of abelian groups with \( C_n = \mathbb{Z}/p^2 \) and \( d_n = p \) for all \( n \in \mathbb{Z} \); i.e., \( C \) has the form

\[
\cdots \to \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \to \cdots
\]

(a) Compute the endomorphism ring \( \text{Hom}_{K(\text{Ab})}(C, C) \) of \( C \) in the homotopy category of chain complexes.

**Solution.** Any chain map \( f: C \to C \) is a collection of maps \( f_n \in \text{Hom}(C_n, C_n) = \text{Hom}(\mathbb{Z}/p^2, \mathbb{Z}/p^2) \approx \mathbb{Z}/p^2 \) such that

\[
pf_n = f_{n-1} p \quad \text{for all } n.
\]

In other words, \( \{f_n\} \) is a sequence of elements in \( \mathbb{Z}/p^2 \) such that \( f_n \equiv f_{n-1} \mod p \) for all \( n \). Write \( \phi(f) \in (\mathbb{Z}/p^2)/p \approx \mathbb{Z}/p \) for this common congruence class.

A chain homotopy \( f \sim f' \) is a collection of maps \( h_n \in \text{Hom}(C_n, C_{n+1}) = \text{Hom}(\mathbb{Z}/p^2, \mathbb{Z}/p^2) \approx \mathbb{Z}/p^2 \) such that

\[
ph_n + h_{n-1} p = f_n - f'_n.
\]

I claim that \( f \sim f' \) iff \( \phi(f) = \phi(f') \). Since \( \phi(f - f') = \phi(f) - \phi(f') \) it is enough to show that \( f \sim 0 \) iff \( \phi(f) = 0 \) (i.e., iff each \( f_n \equiv 0 \mod p \)).

Clearly, if \( f \sim 0 \) by some homotopy \( h \), then \( f_n = p(h_n + h_{n-1}) \equiv 0 \mod p \), so \( \phi(f) = 0 \).

Conversely, if \( \phi(f) = 0 \), construct \( h \) as follows. Set \( h_0 = 0 \). For \( n \geq 1 \), choose \( h_n \in \mathbb{Z}/p^2 \) recursively as a solution to the equation

\[
ph_n = f_n - ph_{n-1} \quad \text{in } \mathbb{Z}/p^2,
\]

which exists because \( f_n \equiv 0 \mod p \). Likewise, for \( n \leq -1 \), choose \( h_n \) recursively as a solution to

\[
ph_n = f_{n+1} - ph_{n+1} \quad \text{in } \mathbb{Z}/p^2.
\]

Thus, \( \phi: \text{Hom}_{K(\text{Ab})}(C, C) \to \mathbb{Z}/p \) is a bijection, and it is immediate that \( \phi \) is a ring homomorphism, i.e., \( \phi(f \circ f') = \phi(f) \phi(f') \).

(b) Show that \( H_* C \approx 0 \), but that \( 0 \to C \) is not a chain homotopy equivalence.

**Solution.** By part (a), the identity map \( 1_C: C \to C \) is not homotopic to \( 0 \). Thus \( 0 \to C \) cannot be a chain homotopy equivalence, since then we would necessarily
have that the composite $C \to 0 \to C$ (which is 0) be homotopic to $1_C$. On the other hand, $H_*C$ since the sequence $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$ is exact.

(4) Given $C, D \in \text{Ch}(\mathcal{A})$, define

$$\text{Hom}(C, D)_n := \prod_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(C_p, D_{p+n}),$$

and define $d: \text{Hom}(C, D)_n \to \text{Hom}(C, D)_{n-1}$ by

$$(df)_p := d_D \circ f_p - (-1)^n f_{p-1} \circ d_C$$

where $f = (f_p) \in \text{Hom}(C, D)_n$.

(a) Show that $\text{Hom}(C, D)$ is a chain complex of abelian groups.

**Solution.** Verify that $d \circ d = 0$. If $f \in \text{Hom}(C, D)_n$, then

$$(ddf)_p = d_D \circ (df)_p - (-1)^n (df)_{p-1} \circ d_C$$

$$= d_D \circ (d_D \circ f_p - (-1)^n f_{p-1} \circ d_C)$$

$$- (-1)^n (d_D \circ f_{p-1} - (-1)^n f_{p-2} \circ d_C) \circ d_C$$

$$= -(-1)^n d_D \circ f_{p-1} \circ d_C - (-1)^n d_D \circ f_{p-1} \circ d_C$$

$$= 0.$$

(5) Fix a chain map $f: B \to C$. We use the constructions $C \xrightarrow{i_C} \text{cone}(f) \xrightarrow{\delta} B[-1]$ described in class.

(a) Describe and prove a bijective correspondence between $\text{Hom}_{\text{Ch}}(\text{cone}(f), D)$ and the set of pairs $(u, h)$ with $u \in \text{Hom}_{\text{Ch}}(C, D)$ and $h$ a chain homotopy $uf \sim 0$.

**Solution.** Recall that $\text{cone}(f)_n = B_{n-1} \oplus C_n$, and $d_{\text{cone}(f)}(x, y) = (-d_B(x), d_C(y) - f(x))$.

Given a pair consisting of $u \in \text{Hom}(C, D)_0$ and $h \in \text{Hom}(B, D)_1$, define $\alpha \in \text{Hom}(\text{cone}(f), D)_0$ by

$$\alpha(x, y) := h(x) + u(y).$$
Consider the domain $\mathbb{R}$.

(a) Show that $I$ is not a principal ideal. (Hint: use that $z \mapsto |z|^2$ defines a multiplicative function $R \to \mathbb{Z}_{\geq 0}$.)

**Solution.** If $I = (z)$, then $|z|^2$ must divide $|2|^2 = 4$ and $|\sqrt{-6}|^2 = 6$, whence $|z|^2 \in \{1, 2\}$. But $|z|^2 = |a + b\sqrt{-6}|^2 = a^2 + 6b^2$ for $a, b \in \mathbb{Z}$, and so the only possibilities are $z \in \{\pm 1\}$. Thus, $I = (z)$ only if it is the unit ideal of $R$. But $\phi: R \to \mathbb{F}_2$ defined by $\phi(a + b\sqrt{-6}) = a$ is a non-trivial ring-homomorphism with $I \subseteq \text{Ker } \phi$.

(b) Show that there is no monomorphism $R \oplus R \to I$ of $R$-modules, and that therefore $I$ cannot be a free module since it is not principal. (Hint: show there are no $R$-module monomorphisms $R \oplus R \to R$.)

**Solution.** An $R$-module map $f: R \oplus R \to R$ is given by $f(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in R$. If $\alpha_1 = \alpha_2 = 0$, then clearly $f$ is not a monomorphism. Otherwise, the identity $f(\alpha_2, -\alpha_1) = 0$ demonstrates that $f$ is not a monomorphism. It follows that $I$ cannot be a free module. It is not isomorphic to 0, and is not isomorphic to $R$ by (a). If $I$ is a free module on at least two generators, then there is a monomorphism $R \oplus R \to I$ determined by the choice of any two distinct basis elements. Thus, we have shown that $I$ cannot be free.

(c) Show that $I$ is projective. (Hint: construct an $R$-module homomorphism $f: I \to R \oplus R$ splitting the evident surjection $\pi: R \oplus R \to I$ (i.e., $\pi f = 1_I$); you can define $f$ by $f(z) = (\alpha z, \beta z)$, where $\alpha, \beta$ are suitably chosen elements of $K$.)

**Solution.** Here $\pi(x, y) = 2x + \sqrt{-6}y$. We define $f: I \to R \oplus R$ by

$$f(x) = (-x, -\sqrt{-6}x),$$

which is well-defined since $2(\sqrt{-6}/2), \sqrt{-6}(\sqrt{-6}/2) \in R$, and we have $\pi f(x) = -2x + 3x = x$. 

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