(1) Describe a preadditive category $C$ with the property that $\text{Fun}^{\text{add}}(C, A) \approx \text{Ch}(A)$ for any preadditive category $A$.

Solution. Let $C$ have object set $\mathbb{Z}$ (the set of integers), and define

$$\text{Hom}_C(m, n) = \begin{cases} \mathbb{Z} & \text{if } n = m, \\ \mathbb{Z} & \text{if } n = m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The notation is meant to imply that $1_m$ and $d_m$ are generators of their respective Hom-groups. Composition is defined so that

$$(b 1_m) \circ (a 1_m) = (ab 1_m), \quad (b d_m) \circ (a 1_m) = (b 1_m) \circ (a d_m) = (ab d_m)$$

and

$$(b d_{m-1}) \circ (a d_m) = 0.$$ 

Composition involving any map named “0” is 0. It is clear that this is a category, and that composition is bilinear, so that it is a pre-additive category.

A chain complex $C$ in $A$ corresponds to the additive functor $F: C \to A$ defined by $F(n) = C_n$, with $F(d_m): m \to m - 1 = d_m^c: C_m \to C_{m-1}$.

(2) Let $R$ be a pre-additive category with a finite set of objects $\{X_1, \ldots, X_n\}$. Let $R := \prod_{i,j=1}^n \text{Hom}_R(X_j, X_i)$, i.e., $R$ is the set of $n^2$-tuples $(r_{ij})_{i,j=1}^n$ where $r_{ij}: X_j \to X_i$ in $R$.

(a) Define a product on $R$, show that your product makes $R$ into an associative ring, and prove that there is an equivalence of categories $R-\text{mod} \approx \text{Fun}^{\text{add}}(R, \text{Ab})$.

Solution. Use the matrix product: if $r = (r_{ij})$ and $s = (s_{ij})$, set $(rs)_{ij} := \sum_k s_{ik} r_{kj}$. This is associative by the usual proof that matrix multiplication is associative, and a unit is given by $1 = (\delta_{ij})$, where $\delta_{ii} = 1_{X_i}$ and $\delta_{ij} = 0$ if $i \neq j$.

Define a functor $\Xi: \text{Fun}^{\text{add}}(R, \text{Ab}) \to R-\text{mod}$, sending a functor $F$ to the module $M_F$ defined by

$$M_F := \prod_{i=1}^n F(X_i), \quad (rm)_i := \sum_j F(r_{ij})(m_j),$$

and on objects sending a natural map $\lambda: F \to F'$ to the map $\lambda := \prod_{i=1}^n \lambda(X_i): M_F \to M_{F'}$, where $\lambda(X_i): F(X_i) \to F'(X_i)$.

Introduce the notation $e_{ij}(f) \in R$ where $f \in \text{Hom}_R(X_j, X_i)$ for the matrix such that $(e_{ij}(f))_{ij} = f$, while all other entries of $e_{ij}(f)$ are 0. Note that $e_{ij}(f) e_{jk}(g) = e_{ik}(fg)$, and that $r = \sum_{ij} e_{ij}(r_{ij})$. Set $e_i := e_i(1_{X_i})$, which satisfies $e_i e_i = e_i$.

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I’ll show that \( \Xi \) is an equivalence, by showing that it is essentially surjective, faithful, and full.

To show that the functor \( \Xi \) is essentially surjective, consider \( e_i \in R \) defined to be the “elementary matrix” with \((e_i)_{i,i} = 1\), and \((e_i)_{j,k} = 0\) if either \( i \neq j \) or \( i \neq k \). Note that \( e_i e_j = e_i \). Given an \( R \)-module \( M \), set
\[
F(X_i) := e_i M \subseteq M.
\]
For \( f \in \text{Hom}_R(X_j, X_i) \), let \( F(f) : F(X_j) \to F(X_i) \) be the map
\[
e_{ij}(f) : e_j M \to e_i M,
\]
where we use the fact that \( e_i e_{ij}(f) = e_{ij}(f) \) to see that the image really lands in \( e_i M \). It is straightforward to show that \( F \) is an additive functor. The evident function \( M \to M_F \) sending \( x \in M \) to \((e_i x) \in \prod e_i M \) is a map of \( R \)-modules (since \( e_i(rx) = \sum e_{ij}(r e_j x) \)), and is a bijection (since \( \sum e_i = 1_R \)). Thus, \( M \) is isomorphic to an object in the image of our functor \( \Xi \).

It is immediate that \( \Xi \) is faithful, for if \( \lambda = 0 \), then each \( \lambda(X_i) = 0 \). To show that \( \Xi \) is full, let \( F, F' : R \to \text{Ab} \) be additive functors, and let \( \mu : M_F \to M_{F'} \) be a map of \( R \)-modules. Because \( M_F = \prod e_i M_F = \prod F(X_i) \) and \( M_{F'} = \prod e_i M_{F'} = \prod F'(X_i) \), we can write \( \mu = \sum e_{ij} e_i \mu e_j \), where \( e_i e_{ij} e_j : F'(X_j) \to F(X_i) \). The key thing to show is that \( e_i \mu e_j = 0 \) when \( i \neq j \), and thus \( \mu = \sum e_{ij} e_i \mu e_j \). This follows because \( \mu \) is a map of \( R \)-modules, so in particular \( e_k \mu e_i = e_k \mu \) for all \( k \).

Thus, we may define \( \lambda : F \to F' \) so that \( \lambda(X_i) = e_i \mu e_i : e_i M_F \to e_i M_{F'} \), and then \( \lambda = \sum e_{ij} e_i \mu e_j \).

(b) Let \( A \) be a ring, and suppose \( \mathcal{R} \) is defined by \( \text{Hom}_R(X_j, X_i) = A \) for all \( 1 \leq i, j \leq n \), with composition defined by multiplication in \( A \). Show that if \( n \geq 1 \), there is an equivalence of categories \( \text{Fun}^{\text{add}}(\mathcal{R}, \text{Ab}) \approx A\text{-mod} \), defined on objects by \( F \mapsto F(X_1) \) and on maps by \( f \mapsto f(X_1) \).

**Solution.** Define a functor \( \Theta : \text{Fun}^{\text{add}}(\mathcal{R}, \text{Ab}) \to A\text{-mod} \) on objects and morphisms by
\[
\Theta F := F(X_1), \quad \Theta f := f(X_1).
\]
It is clear that this defines a functor, and in fact an additive functor.

To show that \( \Theta \) is essentially surjective, consider an \( A \)-module \( M \), and define \( F : \mathcal{R} \to \text{Ab} \) by
\[
F(X_i) := M, \quad F(a)(m) := am.
\]
That is, an element \( a \in \text{Hom}_R(X_j, X_i) = A \) acts by multiplication by \( a \) in the module \( M = F(X_j) = F(X_i) \). I claim that there is an isomorphism \( \Theta F \to M \) of \( A \)-modules. The map is just the identity map \( \Theta F = F(X_1) = M \); it is a map of \( A \)-modules, because \( F(a) \) acts as multiplication by \( a \).

To show that \( \Theta \) is faithful, suppose \( \lambda : F \to F' \) is a natural map with the property that \( \Theta \lambda = 0 \). That is, \( \lambda(X_1) = 0 \). Let \( \delta_{ij} \in \text{Hom}_R(X_j, X_i) = A \) be the element corresponding to the identity element \( 1_A \in A \). The fact that \( \lambda \) is natural means that
\[
F'(f) \lambda(X_j) = \lambda(X_i) F(f)
\]
for all \( f \in \text{Hom}_R(X_j, X_i) \), and in particular holds when \( f = \delta_{ij} \). Each \( \delta_{ij} \) is an
isomorphism in \( R \) with \( \delta_{ij}^{-1} = \delta_{ji} \), so we can write
\[
\lambda(X_j) = F'(\delta_{ji})\lambda(X_i)F(\delta_{ij}).
\]
In particular, \( \lambda(X_1) = 0 \) implies \( \lambda(X_j) = 0 \) for all \( j \), and thus \( \lambda = 0 \).
To show that \( \Theta \) is full, let \( g: F(X_1) \to F(X_j) \) be a map of \( A \)-modules. Define
\[
\lambda(X_j) := F'(\delta_{j1})gF(\delta_{1j}): F'(X_j) \to F'(X_j).
\]
These define a natural transformation \( \lambda: F \to F' \). To see this, let \( f \in \text{Hom}_R(X_j, X_i) = A \), and note that
\[
F'(f)\lambda(X_j) = F'(f)F'(\delta_{j1})gF(\delta_{1j}) = F'(\delta_{j1})gF(\delta_{1j})F(f),
\]
using the facts that \( f\delta_{j1} = \delta_{j1}f \) and \( f\delta_{1j} = \delta_{1j}f \) (these happen in \( A \), and the \( \delta \)s are really the identity element of \( A \)), and the fact that \( g \) is an \( A \)-module map, so
commutes with \( A \)-scalar multiplication. It is then straightforward to check that
\( \lambda \).
(c) Show that \( A \text{-mod} \approx M_{n \times n}(A)\text{-mod} \) for any ring \( A \) and \( n \geq 1 \).

**Solution.** We have defined equivalences \( A \text{-mod} \approx \text{Fun}^{\text{add}}(R, \text{Ab}) \approx M_{n \times n}(A)\text{-mod} \).

(3) Show that the category of finitely generated free abelian groups is an additive category
such that every map has a kernel and a cokernel, but that it is not an abelian category.
(Hint: cokernels are not necessarily quotient groups.)

**Solution.** Write \( F \subset \text{Ab} \) for the full subcategory of finite generated free abelian
groups. It is clearly pre-additive, since \( \text{Hom}_F(A, B) = \text{Hom}_{\text{Ab}}(A, B) \) is an abelian
group, and composition is bilinear.
The direct sum \( A \oplus B \) in \( \text{Ab} \) of two finitely generated free abelian groups is still free
and finitely generated. Furthermore, it is still a direct sum in \( F \). Thus \( F \) is additive.
Given a map \( f: A \to B \) in \( F \), let \( K = \text{Ker} f \), the kernel of \( K \) in \( \text{Ab} \). The group \( K \)
is finitely generated (because \( \mathbb{Z} \) is Noetherian) and is torsion free, and thus free by the
structure theorem for finitely generated abelian groups. It is straightforward to check that
\( K \) is also the kernel of \( f \) in \( F \).

Given a map \( f: A \to B \) in \( F \), let \( C = \text{Cok} f \), the cokernel in \( \text{Ab} \). Clearly \( C \)
is finitely generated, but may fail to be free. Let \( C_t \subseteq C \) be the torsion subgroup
\( C_t = \{ x \in C \mid nx = 0 \text{ for some } n \geq 1 \} \), and let \( \overline{C} := C/C_t \). Then \( \overline{C} \)
is in \( F \) by the structure theorem for finitely generated abelian groups. I claim that the composite
\( B \xrightarrow{p} C \xrightarrow{q} \overline{C} \) is a cokernel in \( F \). To see this, suppose \( g: B \to D \) is a map in \( F \)
such that \( gf = 0 \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{p} \\
D & \xleftarrow{h} & C \\
\end{array}
\]

Then there exists a unique homomorphism \( h: C \to D \) such that \( hp = g \). The image
\( h(C_t) = 0 \) in \( D \) since \( D \) is torsion free, and thus there exists a unique \( h': \overline{C} \to D \)
such that \( h'q = h \), and thus \( h'qp = g \), as desired.
\[ F \] is not abelian: the cokernel of \( 2: \mathbb{Z} \to \mathbb{Z} \) in \( F \) is 0: \( \mathbb{Z} \to 0 \), while the kernel of \( 0: \mathbb{Z} \to 0 \) is \( \mathbb{Z}_2: \mathbb{Z} \to \mathbb{Z} \), which is not equivalent to 2.

(4) Let \( A_1 \) and \( A_2 \) be abelian categories. Show that \( C := A_1 \times A_2 \) is an abelian category, and that the projection and inclusion functors \( A_1 \to C \rightleftharpoons A_2 \) are exact.

**Solution.** The category \( C \) has as objects pairs \((A_1, A_2)\) where \( A_i \) is an object of \( A_i \).

Morphisms \((A_1, A_2) \to (A'_1, A'_2)\) are pairs \((f_1, f_2)\) of maps \( f_i: A_i \to A'_i \) in \( A_i \).

That \( C \) is abelian is entirely straightforward. It is preadditive (add maps by \((f_1, f_2) + (g_2, g_2) := (f_1 + f_2, g_1 + g_2)\)) and additive (direct sums are computed componentwise, so \((A_1, A_2) \oplus (B_1, B_2) \approx (A_1 \oplus B_1, A_2 \oplus B_2)\)). Furthermore, kernels and cokernels exist (and are also computed componentwise), and it is straightforward to show that the axioms for abelian category hold.

The projection functor \( \pi_i: C \to A_i \) is defined on objects by \( \pi_i(A_1, A_2) = A_i \), and on morphisms by \( \pi_i(f_1, f_2) = f_i \). The inclusion functor \( \iota_1: A_1 \to C \) is defined on objects by \( \iota_1(A) = (A, 0) \), and on morphisms by \( \iota_1(f) = (f, 0) \) (and similarly \( \iota_2: A_2 \to C \)). That these functors are exact follows from the fact that all “operations” in \( C \) are performed componentwise.

(5) Let \( R_1, R_2 \) be rings, and let \( R = R_1 \times R_2 \) be the product ring. Show that there is an equivalence of categories \( R-\text{mod} \approx R_1-\text{mod} \times R_2-\text{mod} \).

**Solution.** Remark: it should be noted that this problem is actually a very special case of exercise (2a).

Let \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) in \( R \). For \( M \in R-\text{mod} \), define \( F(M) = (F_1(M), F_2(M)) \) by \( F_i(M) := e_iM \), where \( e_1M \subseteq M \) is made into an \( R_1 \)-module by \( r \cdot x := (r, 0)x \), and \( e_2M \) is made into an \( R_2 \)-module by \( r \cdot x := (0, r)x \). This works because \( e_1(r, 0)x = (r, 0)x \) and \( e_2(0, r)x = (0, r)x \).

Make this a functor by sending an \( R \)-module homomorphism \( f: M \to M' \) to the pair \( F(f) := (f_1, f_2) \), where \( f_i \) is the restriction of \( f \) to \( e_iM \). Note that \( e_iFe_i(x) = feie_i(x) = fe_i(x) \), so in fact \( f(e_iM) \subseteq e_iM \).

That \( F \) is essentially surjective is as follows: given \( R_i \)-modules \( N_i \), set \( M := N_1 \times N_2 \) with \( R \)-module structure given by \( (r_1, r_2) \cdot (x_1, x_2) = (r_1x_1, r_2x_2) \). Then there are evident isomorphisms \( e_iM \iso N_i \), and thus \( F(M) \iso (N_1, N_2) \).

That \( F \) is faithful is as follows: if \( F(f) = (f_1, f_2) = 0 \), then \( f = f_1 + f_2 = 0 \).

That \( F \) is full is as follows: given \( R \)-modules \( M \) and \( M' \), and maps \( g_i: e_iM \to e_iM' \) of \( R_i \)-modules for \( i = 1, 2 \), define \( f: M \to M' \) by \( f(x) = g_1(e_1x) + g_2(e_2x) \). It is clear that the restriction of \( f \) to \( e_iM \) is equal to \( g_i \), since \( e_ie_j = 0 \) when \( i \neq j \).

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