Monday 17 November

1. Group extensions

Now we want to describe low dimensional group cohomology a little more carefully. In particular, we’ll identify $H^2(G; M)$ with a certain group of extensions of $G$ by $M$.

1.1. The 2-groupoid of extensions. An extension of a group $G$ by an abelian group $M$ is an exact sequence of groups

$$0 \to M \to X \to G \to 1.$$ 

We run into trouble because we want to write the (nonabelian) group multiplicatively, but the abelian group $M$ additively. I’ll use the following convention. For an element $m \in M$, I write $[m] \in M$ for the exact same element. But I combine elements $m$ with the “additive” group law, and combine elements $[m]$ with the “multiplicative” group law. Thus

$$[m + m'] = [m][m'], \quad [0] = e.$$ 

If $M$ sits in an extension as above, then $M$ is stable by conjugation by elements in $X$, and this action factors through the quotient group $G$; i.e., $M$ becomes a $ZG$-module. Given $x \in X$, write $\bar{x} \in G$ for its image. We use the notation

$$x[m]x^{-1} = [\bar{x} \cdot m]$$

to represent the conjugation action of $G$ on $M$, which $M$ into a left $ZG$-module.

Given a group $G$ and a left $ZG$-module $M$, we want to classify extensions of $G$ by $M$. We package such extensions into a category. Thus, we have a category $\mathcal{E}(G, M)$, whose objects are exact sequences of groups

$$0 \to M \to X \to G \to 1,$$

such the induced conjugation action by $G$ on $M$ coincides with the given $ZG$-module structure. Morphisms are commutative diagrams

$$\begin{CD}
0 @>>> M @>>> X @>>> G @>>> 1 \\
\quad @V{1_M}VV \quad @V{f}VV \quad @V{1_G}VV \\
0 @>>> M @>>> X' @>>> G @>>> 1
\end{CD}$$

Note that $f$ is necessarily an isomorphism.

Suppose we have two extensions (with groups $X$ and $X'$), and two maps $f, f': X \to X'$ between them. Note that $f'(x)f(x)^{-1} \in M$, and that the value of $f'(x)f(x)^{-1}$ only depends on $\pi(x) \in G$. Thus we can write

$$f'(x) = [d(\bar{x})]f(x)$$

for some function $d: G \to M$. We observe that

$$f'(xy) = [d(\bar{x})][f(x)][d(\bar{y})]f(y) = [d(\bar{x})][f(x)][d(\bar{y})][f(x)^{-1}]f(xy) = [d(\bar{x}) + \bar{x} \cdot d(\bar{y})]f(xy).$$
That is, $d: G \to M$ is a derivation, i.e., a function satisfying
\[ d(gh) = d(g) + g \cdot d(h). \]
So, each homset $\text{Hom}_{E(G,M)}(X, X')$ can be put in bijective correspondence with the set $\text{Der}(G, M)$ of derivations. (This isn’t canonical, but depends on a choice of “basepoint” $f \in \text{Hom}_{E(G,M)}(X, X')$.)

There is a distinguished split extension $0 \to M \to M \times G \to G \to 1$, with product given by $(m, g)(m', g') := (m + g \cdot m', g'g)$. An extension $X$ is isomorphic to the split extension if and only if $\pi: X \to G$ admits a group homomorphism section. Note that $\text{Hom}_{E(G,M)}(M \times G, M \times G) \approx \text{Der}(G, M)$ canonically, so that $d \in \text{Der}(G, M)$ corresponds to $f: M \times g \to M \times G$ defined by $f(m, g) = (m + d(g), g)$.

There is one more piece of information to consider. Given maps $f, f': X \to X'$ of extensions, and $u \in M$, we write
\[ u: f \Rightarrow f' \]
if
\[ f'(x) = uf(x)u^{-1} = [u]f(x)[u]. \]
We can think of this as producing a category, whose objects are the homset $\text{Hom}_{E(G,M)}(X, X')$, and so that morphisms $f \Rightarrow f'$ are $u \in M$ such that the above identity holds. The composite of $f \Rightarrow f' \Rightarrow f''$ is the element $u' + u: f \Rightarrow f''$. Every such object is invertible: the inverse of $u$ is $u^{-1}$. Write $\text{Hom}(X, X')$ for this category.

An easy exercise shows that given $u: f \Rightarrow f'$, then
\[ f'(x) = [u]f(x)[-u] = [u]f(x)[-u]f(x)^{-1}f(x) = [u - u \cdot u]f(x). \]
That is, $f'(x) = [D_u(x)]f(x)$ where $D_u$ is an inner derivation, given by $D_u(g) = u - g \cdot u$. We write $\text{InnDer}(G, M) \subset \text{Der}(G, M)$ for the subgroup of inner derivations.

The whole structure $E(G, M)$ here is a 2-groupoid. We have

- Objects $X$. Extensions of $G$ by $M$.
- 1-Morphisms $f: X \to X'$. Maps of extensions.
- 2-Morphisms $u: f \Rightarrow f'$. Elements $u \in M$ such that $f' = [D_u]f$.

Composition of 1-morphisms extends to a functor of categories
\[ \text{Hom}(X', X'') \times \text{Hom}(X, X') \to \text{Hom}(X, X''). \]
There is a Baer sum on extensions: define $X * X'$ to be the quotient of the group $X \times_G X'$ obtained by identifying $(m, 0) \sim (0, m) \in M \times M \subseteq X \times_G X'$ for all $m \in M$.

1.2. Exercise. $(M \times G) * X \approx X \approx X \times (M \times G)$.

1.3. Remark. The Baer sum gives $E(G, M)$ the structure of what is known as a symmetric monoidal 2-groupoid, or picard 2-groupoid.

We can extract three abelian groups from this.

- $\pi_0 E(G, M) :=$ isomorphism classes of objects of $E(G, M)$, with group structure given by Baer sum.
- $\pi_1 E(G, M) :=$ isomorphism classes of objects of $\text{Hom}(M \times G, M \times G)$; i.e., $\text{Der}(G, M)/\text{InnDer}(G, M)$, with group structure given by addition of derivations.
- $\pi_2 E(G, M) :=$ the group of 2-automorphisms of the trivial derivation $0: G \to M$, which are exactly the $u \in M$ such that $D_u = 0$, i.e., $u \in M^G$.

1.4. Theorem. Let $G$ be a group and $M$ a left $\mathbb{Z}G$-module.

- $H^0(G; M) \approx \pi_2 E(G, M) \approx M^G$.
- $H^1(G; M) \approx \pi_1 E(G, M) \approx \text{Der}(G, M)/\text{InnDer}(G, M)$.
- $H^2(G; M) \approx \pi_0 E(G, M)$, the set of isomorphism classes of extensions.
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