Monday, 6 October

1. Hom and Tensor

1.1. Hom. Given $M, N \in R\text{-mod}$, we have an abelian group $\text{Hom}_{R\text{-mod}}(M, N)$, the set of left $R$-module homomorphisms $M \to N$. Explicitly, it is a subset of the set of all abelian group maps $f: M \to N$ such that $f(rx) = rf(x)$. Similarly, for right modules $M', N' \in \text{mod-R}$, we have $\text{Hom}_{\text{mod-R}}(M', N')$, the set of right $R$-module homomorphisms.

Note: I might call either of these $\text{Hom}_R$, which would not cause confusion here. But if there are many module structures around, it might.

Recall that a right $R$-module structure is the same thing as a left $R^{\text{op}}$-module structure (by $r \cdot m := mr$, which must satisfy $(rr') \cdot m = m(rr') = (mr)r' = r' \cdot (mr) = r' \cdot (r \cdot m)$). In fact, this defines an equivalence of categories $\text{mod-R} \to R^{\text{op}}\text{-mod}$.

The Hom-construction defines a functor $R\text{-mod}^{\text{op}} \times R\text{-mod} \to \text{Ab}$. It is not an additive functor; rather, it is additive in each variable separately. We do obtain functors $\text{Hom}_{R\text{-mod}}(M, -): R\text{-mod} \to \text{Ab}$ and $\text{Hom}_{R\text{-mod}}(-, N): R\text{-mod}^{\text{op}} \to \text{Ab}$. We recall

1.2. Proposition.

- The functor $\text{Hom}_{R\text{-mod}}(M, -): R\text{-mod} \to \text{Ab}$ is left exact and preserves arbitrary products in $R\text{-mod}$. That is, it takes kernels/products in $R\text{-mod}$ to kernels/products in $\text{Ab}$. It is exact if and only if $M$ is projective.
- The functor $\text{Hom}_{R\text{-mod}}(-, N): R\text{-mod}^{\text{op}} \to \text{Ab}$ is left exact and preserves arbitrary products in $R\text{-mod}^{\text{op}}$. That is, it takes cokernels/coproducts in $R\text{-mod}$ to kernels/products in $\text{Ab}$. It is exact if and only if $N$ is injective.

1.3. Bimodules. An $R$-$S$-bimodule is an $M$ which is both a left $R$-module and a right $S$-module, so that the module structures commute. That is, $(rm)s = r(ms)$. We write $R\text{-mod-S}$ for the category of $R$-$S$-bimodules.

Suppose $M \in R\text{-mod-S}$ and $N \in R\text{-mod-T}$. Then

$\text{Hom}_{R\text{-mod}}(M, N)$

inherits a natural structure of $S$-$T$-bimodule, given by

$(sf)(m) := f(ms), \quad (ft)(m) := f(m)t$.

Date: October 8, 2014.
1.4. **Tensor product.** Given $M \in \text{mod-}R$ and $N \in R\text{-mod}$, an $R$-bilinear map to an abelian group $A$ is a function $\beta: M \times N \to A$ such that

$$\beta(x + x', y) = \beta(x, y) + \beta(x', y), \quad \beta(x, y + y') = \beta(x, y) + \beta(x, y'), \quad \beta(xr, y) = \beta(xy, r).$$

Note that an $R$-bilinear map is not a homomorphism of abelian groups (which would mean $\beta(x + x', y + y') = \beta(x, y) + \beta(x', y')$).

The tensor product $T$ of $M$ and $N$ is defined as follows. Let $\mathbb{Z}[M \times N]$ denote the free abelian group on the set $M \times N$; a typical element in $R[\mathbb{Z}[M \times N]]$ is $\sum_{k=1}^n a_k (x_k, y_k)$, with $a_k \in R$, $(x_k, y_k) \in M \times N$. The tensor product is the quotient

$$T := \mathbb{Z}[M \times N]/B,$$

where $B$ is the subgroup generated by

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'), \quad (xr, y) - (x, ry), \quad (x + x') \otimes y - x \otimes y - x' \otimes y, \quad x \otimes (y + y') - x \otimes y - x \otimes y', \quad (xr) \otimes y - x \otimes (ry).$$

where $x, x' \in M$, $y, y' \in N$, and $r \in R$. Write $[x, y] \in T$ for the equivalence class of $(x, y)$.\[\square\]

1.5. **Proposition.** The function $\alpha: M \times N \to T$ defined by $\alpha(x, y) := [x, y]$ is bilinear, and is the universal bilinear map from $M \times N$, in the sense that given a bilinear $\beta: M \times N \to A$, there exists a unique group homomorphism $f: T \to A$ such that $f \alpha = \beta$.

**Proof.** This is basically a tautology.

Typically, we write $M \otimes_R N := T$ for the tensor product, and write $x \otimes y$ for the element $[x, y] \in T$. It is important to note that elements of $T$ are represented as finite linear combinations of $x \otimes y = [x, y]$, and such representations are not unique. Thus, given $u \in M \otimes_R N$, the best we can say is that

$$u = \sum_{k=1}^n x_i \otimes y_i \text{ for some } n \geq 0, x_i \in M, y_i \in M, i = 1, \ldots, n.$$

Two such elements $u, u'$ are equal if and only if the expression $\sum_i x_i \otimes y_i - \sum_j x'_j \otimes y'_j$ can be rewritten formally as a $\mathbb{Z}$-linear combination of the basic relations

$$(x + x') \otimes y - x \otimes y - x' \otimes y, \quad x \otimes (y + y') - x \otimes y - x \otimes y', \quad (xr) \otimes y - x \otimes (ry).$$

In other words, tensor products typically do not come with a convenient enumeration of their elements (unlike cartesian products). Instead, they are given in terms of a presentation by generators and relations.

Because of its universal propery, tensor product defines a functor $\text{mod-}R \times R\text{-mod} \to \text{Ab}$. It is not an additive functor, but is rather additive in each variable; i.e., we get additive functors

$$M \otimes_R - : R\text{-mod} \to \text{Ab}, \quad - \otimes_R N : R\text{-mod} \to \text{Ab}.$$

1.6. **Proposition.**

- The functor $M \otimes_R - : R\text{-mod} \to \text{Ab}$ is right exact and preserves arbitrary coproducts. That is, it takes cokernels/coproducts in $R\text{-mod}$ to cokernels/coproducts in $\text{Ab}$.

- The functor $- \otimes_R N : R\text{-mod} \to \text{Ab}$ is right exact and preserves arbitrary coproducts. That is, it takes cokernels/coproducts in $\text{mod-}R$ to cokernels/coproducts in $\text{Ab}$.

The key step of the proof is to properly understand what it means to “preserve” cokernels or coproducts. In general, if $F: A \to B$ is a functor between categories (not necessarily additive), and $A$ and $B$ have coproducts, then for $A = \{A_i\}_{i \in S}$ in $A^S$ there is a map

$$\bigoplus_{i \in S} F(A_i) \xrightarrow{\cong} F(\bigoplus_{i \in S} A_i),$$
characterized by the property that the composite of
\[
F(A_i) \xrightarrow{\gamma_{\{A_i\}}^{(F(A_i))}} \bigoplus_{i \in S} F(A_i) \xrightarrow{\subseteq} F\left( \bigoplus_{i \in S} A_i \right)
\]
is equal to \(F(\gamma_{\{A_i\}})\). In the proposition, \(F\) represents the functors \(M \otimes_- R\) or \(- \otimes R N\).

Likewise, if \(F: A \to B\) is an additive functor between abelian categories, and \(f: A' \to A\) is a map in \(A\), we use
\[
c: \text{Cok}(F(f)) \to F(\text{Cok} f)
\]
which is the unique map making the diagram
\[
\begin{array}{ccc}
F(A') & \xrightarrow{F(f)} & F(A) \\
\downarrow & & \downarrow c \\
\text{Cok}(F(f)) & \rightarrow & F(\text{Cok} f)
\end{array}
\]
commute.

Department of Mathematics, University of Illinois, Urbana, IL

E-mail address: rezk@math.uiuc.edu