Friday, 12 September

Say $A$ has **enough projectives** if for every object $M$ there exists an epimorphism $P \to M$ with $P$ projective.

$R$-modules have enough projectives. For instance, for any $M$, the map

$$\bigoplus_{x \in M} Re_x \to M, \quad \sum r_i e_{x_i} \mapsto \sum r_i x_i$$

is an epimorphism from a free module.

0.1. **Resolutions.** Given $M$ in $A$, a (left) resolution of $M$ is a chain complex $C$ in $\text{Ch}(A)$ with $C_n = 0$ for $n < 0$, together with a map $\epsilon: C_0 \to M$ making

$$\cdots \to C_2 \to C_1 \xrightarrow{d_{C,1}} C_0 \xrightarrow{\epsilon} M \to 0$$

an exact sequence.

We will identify regard $A$ with the full subcategory of $\text{Ch}(A)$ consisting of chain complexes $C$ with $C_n = 0$ for $n \geq 0$. A chain map $C \to D$ between two such complexes is exactly described by the a map $C_0 \to D_0$ of objects in $A$. Thus, we have a fully faithful functor $A \to \text{Ch}(A)$. I will often use this inclusion functor without comment.

Also note that any chain homotopy involving chain complexes concentrated in degree 0 satisfies $h = 0$, and thus the composite $A \to \text{Ch}(A) \to \text{K}(A)$ is also fully faithful. That is, $A$ can be identified with a full subcategory of the homotopy category of chain complexes.

Under this identification $A \subseteq \text{Ch}(A)$, a left-resolution of $M$ is exactly a quasi-isomorphism

$$\epsilon: C \to M$$

where $C \in \text{Ch}_{\geq 0}(A)$. This is how I will usually write it.

We say that $P \in \text{Ch}_{\geq 0}(A)$ is a complex of projectives if each $P_n$ is projective. (Warning: This does not imply that $P$ is a projective object in $\text{Ch}(A)$.)

0.2. **Exercise.** What are the projective objects of $\text{Ch}(A)$?

We say $\epsilon: P \to M$ is a **projective resolution** if $P$ is a complex of projectives and $\epsilon$ is a resolution.

0.3. **Proposition.** If $A$ has enough projectives, then every object of $A$ admits a projective resolution.

**Proof.** Easy. Given $M \in A$, choose epi $\epsilon: P_0 \to M$; choose epi $\tilde{d}: P_1 \to \text{Ker} \epsilon$ and set $d_{P,1} = (\text{ker} \epsilon)\tilde{d}: P_1 \to P_0$; inductively choose epi $\tilde{d}: P_n \to \text{Ker} d_{P,n-1}$ and set $d_{P,n} = (\text{ker} d_{P,n-1})\tilde{d}$. \hfill $\square$

0.4. **Lemma** (Key lemma). Let $M \in A$. Let $\eta: Q \to M$ be a resolution, let $\epsilon: P \to M$ be a map of complexes, and suppose that $P$ is a complex of projectives and $P_n = 0$ for $n < 0$. Then there exists

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a chain map \( f : P \to Q \) such that \( \eta f = \epsilon \); furthermore, such \( f \) is unique up to chain homotopy.

\[
\begin{array}{c}
Q \\
\downarrow \eta \\
\end{array}
\quad
\begin{array}{c}
P \\
\downarrow \epsilon \\
M \\
\end{array}
\]

**Proof.** Usual proof. Emphasise the use of the fact that if \( P \) is a projective object in \( \mathcal{A} \) then \( \text{Hom}(P,-) \) is exact.

Thus, we construct \( f_n \) inductively, given \( f_k \) for \( k < n \) such that \( \text{d}_Q f_{n-1} = f_{n-2} \text{d}_P \), using the exact sequence

\[
\text{Hom}(P_n, Q_n) \xrightarrow{\text{d}_Q} \text{Hom}(P_n, Q_{n-1}) \xrightarrow{\text{d}_Q} \text{Hom}(P_n, Q_{n-2})
\]

\[
f_n - - - - - - \rightarrow \text{d}_Q f_n = f_{n-1} \text{d}_P \rightarrow \text{d}_Q f_{n-1} \text{d}_P = f_{n-1} \text{d}_P \text{d}_P = 0
\]

Given \( f, f' \) such that \( \eta f = \epsilon = \eta f' \), build a chain homotopy by iteratively solving \( \text{d}_Q h_n = f_n - f'_n - h_{n-1} \text{d}_Q \) for \( h_n \) using

\[
\text{Hom}(P_n, Q_{n+1}) \xrightarrow{\text{d}_Q} \text{Hom}(P_n, Q_n) \xrightarrow{\text{d}_Q} \text{Hom}(P_n, Q_{n-1})
\]

\[
h_n - - - - - - \rightarrow \text{d}_Q h_n = f_n - f'_n - h_{n-1} \text{d}_Q \rightarrow \text{d}_Q (f_n - f'_n - h_{n-1} \text{d}_Q) = 0
\]

since \( \text{d}_Q (f_n - f'_n - h_{n-1} \text{d}_Q) = \text{d}_Q (f_n - f'_n) - (f_n - f'_n) \text{d}_Q + h_{n-1} \text{d}_Q \text{d}_Q = 0 \). \( \square \)

Consequences:

- Given a projective resolution \( \epsilon : P \to M \), any chain map \( f : P \to M \) such that \( f \epsilon = \epsilon \) is chain homotopic to \( 1_P \).
- Any two projective resolutions of an object are chain homotopy equivalent.
- Given projective resolutions \( \epsilon : P \to M \), \( \eta : Q \to N \), and a map \( f : M \to N \), there exists a chain map \( g : P \to Q \) such that \( \eta g = f \epsilon \).
- Furthermore, the chain map \( f \) with such properties is unique up to chain homotopy.
- Using the notation of the previous statement, we have a bijection

\[
\text{Hom}_{\mathcal{K}(\mathcal{A})}(P, Q) \approx \text{Hom}_{\mathcal{A}}(M, N).
\]

0.5. **Remark.** Note that if \( P \in \text{Ch}_{\geq 0}(\mathcal{A}) \) is a bounded below chain complex with \( H_n(P) \approx 0 \) for \( n \neq 0 \), then \( P \) is tautologically a resolution of \( H_0(P) \).

Let \( \mathcal{P}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A}) \) denote the full subcategory of the homotopy category \( \mathcal{K}(\mathcal{A}) \) consisting of chain complexes \( P \) which are degree-wise projective, with \( P_n = 0 \) for \( n < 0 \) and \( H_n(P) \approx 0 \) for \( n \neq 0 \). If \( \mathcal{A} \) has enough projectives, then

\[
H_0 : \mathcal{P}(\mathcal{A}) \to \mathcal{A}
\]

is an equivalence of categories.

Thus, we have two different subcategories of \( \mathcal{K}(\mathcal{A}) \) which are equivalent to \( \mathcal{A} \). Classical homological algebra basically amounts to the game: “replace \( \mathcal{A} \) with \( \mathcal{P}(\mathcal{A}) \) and see what happens”.

0.6. **Exercise.** Let \( \epsilon : P \to M \) be a projective resolution. Show that \( \epsilon \) is a chain homotopy equivalence if and only if \( M \) is a projective object in \( \mathcal{A} \).
0.7. **Examples.** If \( \mathcal{A} \) has the property that every object is projective, then every object is a projective resolution of itself.

0.8. **Example.** If \( R \) is a field, or a division ring, then every object of \( R\text{-mod} \) is projective. This is true more generally of semi-simple rings.

## 1. **Left derived functors**

Let \( F: \mathcal{A} \to \mathcal{B} \) be an additive functor between two abelian categories, and assume that \( \mathcal{A} \) has enough projectives. We associate **left derived functors**

\[
L_i F: \mathcal{A} \to \mathcal{B}, \quad i \geq 0
\]

as follows. First, for each \( M \) in \( \mathcal{A} \) make a choice of projective resolution \( P \to M \). Then define

\[
(L_i F)(M) := H_i(F(P)).
\]

This is evidently a functor, using the fact that any \( f: M \to N \) lifts to a chain map \( \tilde{f}: P \to Q \) between projective resolutions, uniquely up to chain homotopy.

In particular, note that if two maps \( f, g: P \to Q \) are related by a chain homotopy, then so are \( F(f), F(g): F(P) \to F(Q) \). That is, since \( ds + sd = f - g \), then \( F(d)F(s) + F(s)F(d) = F(f) - F(g) \); this uses in a strong way the fact that \( F \) is additive.

Furthermore, each \( L_i F \) is an additive functor. Given \( f, g: M \to N \) with lifts \( \tilde{f}, \tilde{g}: P \to Q \) to projective resolutions, check that \( \tilde{f} + \tilde{g} \) is a lift of \( f + g \), and thus \( H_i F(\tilde{f} + \tilde{g}) = H_i F(\tilde{f}) + H_i F(\tilde{g}) \) since both \( F \) and \( H_i \) are additive.

Note that if we chose a different projective resolution \( P' \to M \), we get a canonical isomorphism \( H_* F(P) \approx H_* F(P') \), induced by any chain map \( P \to P' \) covering \( 1_M \), so it does not matter which resolution we chose.

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