Monday, 25 August

1. Basic notions for abelian Categories

Abelian categories are ones which share some of the basic features of categories of modules over a
ring. In particular, they come with an adequate notion of “kernel” and “cokernel” of a map, which
allows you to talk about exact sequences.

1.1. Pre-additive categories. A pre-additive category (or Ab-category, or category en-
riched over Ab) is a category $\mathcal{A}$, such that

- for every pair of objects $A$ and $B$, the hom-set $\text{Hom}_A(A,B)$ is equipped with the structure
  of abelian group, so that
- $g(f + f') = gf + g'f$, $(g + g')f = gf + g'f$ for $f, f' \in \text{Hom}(A, B)$ and $g, g' \in \text{Hom}(B, C)$.

We write $1_B \in \text{Hom}(B, B)$ for the identity map of $B$, in which case we have $1_B f = f$ and $g 1_B = g$.

The zero map $0 \in \text{Hom}(A, B)$ exists for each pair of objects $A, B$; we use the same symbol “0”
for all of these maps, though they are distinct. Note that $f \circ 0 = 0$ and $0 \circ g = 0$ whenever these are
defined.

Because composition is bilinear, we can express it in terms of tensor products:

$$g \otimes f \mapsto gf : \text{Hom}(B, C) \otimes \text{Hom}(A, B) \to \text{Hom}(A, C).$$

1.2. Example. A pre-additive category with one object is exactly the same thing as an associative
ring (with unit).

1.3. Example. The categories $R$–mod and $\text{mod}$–$R$ of left and right modules over a ring $R$: the set
$\text{Hom}_R(M, N)$ of module maps has an evident abelian group structure.

Note: if $R$ is commutative, then $\text{Hom}_R(M, N)$ admits the structure of an $R$-module. In general,
it is only a module over the center of $R$.

1.4. Example. Given pre-additive $\mathcal{A}$, let $\mathcal{A}^{[1]}$ be the category of arrows in $\mathcal{A}$. That is,
- objects of $\mathcal{A}^{[1]}$ are morphisms $m: M \to M'$ in $\mathcal{A}$,
- morphisms of $\mathcal{A}^{[1]}$ are squares

$$\begin{array}{c}
M \\ m \downarrow \\
\downarrow n \\
M' \\ f' \rightarrow N'
\end{array}
\quad \text{in } \mathcal{A} \text{ such that } nf = f'm.
$$

Exercise: describe the abelian group structure on $\text{Hom}_{\mathcal{A}^{[1]}}$, and show that $\mathcal{A}^{[1]}$ is pre-additive.

1.5. Example. More generally, if $\mathcal{I}$ is a small category and $\mathcal{A}$ is preadditive, the category $\text{Fun}(\mathcal{I}, A)$
is preadditive.

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1.6. Example. Given a pre-additive $\mathcal{A}$, let $\text{Ch}(\mathcal{A})$ be the category of chain complex of objects in $\mathcal{A}$. The objects $C$ of $\text{Ch}(\mathcal{A})$ are collections $(C_n, d_n : C_n \to C_{n-1})_{n \in \mathbb{Z}}$ such that $d_{n-1}d_n = 0$. Morphisms $C \to D$ are collections of maps $(f_n : C_n \to D_n)_{n \in \mathbb{Z}}$ such that $d_nf_n = f_{n-1}d_{n-1}$. Exercise: describe the abelian group structure on $\text{Hom}_{\text{Ch}(\mathcal{A})}$, and show that $\text{Ch}(\mathcal{A})$ is a pre-additive.

1.7. Additive functors. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories. An additive functor (or $\text{Ab}$-functor) is

- a functor $F : \mathcal{A} \to \mathcal{B}$ of categories, such that
- for every pair of objects $M, N \in \mathcal{A}$, the map $F : \text{Hom}_\mathcal{A}(M, N) \to \text{Hom}_\mathcal{B}(F(M), F(N))$ is an homomorphism of groups.

We write $\text{Fun}^{\text{add}}(\mathcal{A}, \mathcal{B})$ for the collection of additive functors; this forms a category, where a map $\lambda : F \to G$ is a natural transformation of functors.

1.8. Example. Let $R$ be an associative ring, and let $\mathcal{A}$ be the pre-additive category with one object $X$ and $\text{Hom}_\mathcal{A}(X, X) = R$. Then

$$\text{Fun}^{\text{add}}(\mathcal{A}, \text{Ab}) = R\text{-mod}.$$ 

In general, we can think of $\text{Fun}^{\text{add}}(\mathcal{A}, \text{Ab})$ as a category of “left-modules” for the pre-additive category $\mathcal{A}$.

1.9. Exercise. Describe a preadditive category $\mathcal{C}$ with the property that $\text{Fun}^{\text{add}}(\mathcal{C}, \mathcal{A}) \cong \text{Ch}(\mathcal{A})$ for any preadditive category $\mathcal{A}$.

1.10. Exercise. Let $\mathcal{R}$ be a pre-additive category with a finite set of objects $\{X_1, \ldots, X_n\}$. Let $R := \prod_{i,j=1}^n \text{Hom}_\mathcal{R}(X_j, X_i)$, i.e., $R$ is the set of $n^2$-tuples $(r_{ij})_{i,j=1}^n$ where $r_{ij} : X_j \to X_i$ in $\mathcal{R}$.

1. Define a product on $R$, show that your product makes $R$ into an associative ring, and prove that there is an equivalence of categories $R\text{-mod} \cong \text{Fun}^{\text{add}}(\mathcal{R}, \text{Ab})$.

2. Let $A$ be a ring, and suppose $\mathcal{R}$ is defined by $\text{Hom}_\mathcal{R}(X_j, X_i) = A$ for all $1 \leq i, j \leq n$, with composition defined by multiplication in $A$. Show that if $n \geq 1$, there is an equivalence of categories $\text{Fun}^{\text{add}}(\mathcal{R}, \text{Ab}) \cong A\text{-mod}$, defined on objects by $F \mapsto F(X_1)$ and on maps by $f \mapsto f(X_1)$.

3. Show that $A\text{-mod} \cong M_{n\times n}(A)\text{-mod}$ for any ring $A$ and $n \geq 1$.

1.11. Exercise. Consider the construction $\text{Ab}^{[1]} \to \text{Ab}$ which sends the object $m : M \to M'$ in $\text{Ab}^{[1]}$ to the object $\text{Ker}(m)$ in $\text{Ab}$. Show that $\text{Ker}$ is an additive functor.

1.12. Exercise. Let $k$ be a field. Show that $F : k\text{-mod} \to k\text{-mod}$ defined by $F(V) = V \otimes_k V$ is a functor which is not additive.

Warning: “Most” functors are non-additive. It will be easy to forget this during this course, since most functors we consider will be additive.

1.13. Exercise. Let $\mathcal{I}$ be a small category, and let $R$ be a ring. Construct a small pre-additive category $\mathcal{A}$ with $\text{ob} \mathcal{A} = \text{ob} \mathcal{I}$ and with the property that $\text{Fun}^{\text{add}}(\mathcal{A}, \text{Ab}) \cong \text{Fun}(\mathcal{I}, R\text{-mod})$.


1.15. Proposition. Let $Z$ be an object in a pre-additive category $\mathcal{A}$. The following are equivalent.

1. For all $X$ in $\mathcal{A}$, $\text{Hom}(Z, X) = \{0\}$.
2. For all $X$ in $\mathcal{A}$, $\text{Hom}(X, Z) = \{0\}$.

We call such an object $Z$, if it exists, a zero-object. Any two zero-objects are isomorphic by a unique isomorphism. We write $0$ for some chosen example of a zero-object.
1.16. **Products and sums.** Recall the definition of product \((P,p : P \to X, p_Y : P \to Y)\) and coproduct \((C,i_X : X \to C, i_Y : Y \to C)\) in a category.

1.17. **Proposition.** Let \(A\) be a pre-additive category. If \((P,p : P \to X, p_Y : P \to Y)\) is a product of \(X\) and \(Y\), then \((P,i_X : X \to P, i_Y : Y \to P)\) is a coproduct, where \(i_X\) and \(i_Y\) are the unique maps such that
\[
(p_X i_X, p_Y i_X) = (1_X, 0), \quad (p_X i_Y, p_Y i_Y) = (0, 1_Y).
\]
Likewise, all coproducts are also products.

**Proof.** Verify and use the identity \(i_X p_X + i_Y p_Y = 1_P\). □

We write \(X \times Y\) for a choice of product, and \(X \oplus Y\) for a choice of coproduct, and note a canonical isomorphism \(X \oplus Y \approx X \times Y\).

1.18. **Additive category.** An additive category is a pre-additive category with a zero-objects and binary products (and coproducts) of all pairs of objects.

1.19. **Remark.** In an additive category, the abelian group structure on Hom-sets is determined by the underlying category. Given \(f, g \in \text{Hom}(X, Y)\), the map \(f + g\) is the composite
\[
X \xrightarrow{(1_X, 1_X)} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\alpha^{-1}} Y \oplus Y \xrightarrow{(1_Y, 1_Y)} Y,
\]
where \(\alpha : Y \oplus Y \to Y \times Y\) is the canonical isomorphism. (Hint: \((1_Y, 1_Y) \alpha^{-1} = p_1 + p_2\).)

This means that if \(A\) and \(B\) are additive, then a functor \(F : A \to B\) is additive if and only if it preserves zero-objects and finite sums/products.

1.20. **Exercise.** Let \(F : A \to B\) be a functor which is an equivalence of categories. Show that if \(A\) is an additive category, then \(B\) is also an additive category and \(F\) is an additive functor.

1.21. **Subobjects and quotient objects.** Recall the definition of monomorphism and epimorphism in an arbitrary category \(C\). Note that these are not synonymous with “injection” and “surjection” even when these make sense.

Let \(f : A \to B\) in a category \(C\). Observe that

- \(f\) is a monomorphism if and only if
  \[
  u \mapsto fu : \text{Hom}(X, A) \to \text{Hom}(X, B)
  \]
  is injective for all \(X\) in \(C\).

- \(f\) is an epimorphism if and only if
  \[
  v \mapsto vf : \text{Hom}(B, Y) \to \text{Hom}(A, Y)
  \]
  is injective for all \(Y\) in \(C\).

Note: notions like monomorphism and epimorphism (or product and coproduct) are defined relative to an ambient category.

1.22. **Exercise.** Let \(A\) be the category of finitely generated free abelian groups and homomorphisms between them. Show that in \(A\), the map \(f : \mathbb{Z} \to \mathbb{Z}\) defined by \(f(x) = 2x\) is both a monomorphism and an epimorphism in \(A\).