Due in class F 5 Dec.

(1) Let $R$ be a domain and $a \in R$. Show that
   (a) $a = 0$ if and only if $Ra = \{0\}$.
   (b) $a \in R^\times$ if and only if $Ra = R$.
   (c) $a$ is reducible in $R$ iff (i) $Ra \neq \{0\}$ and $Ra \neq R$, and (ii) there exists $b \in R$ such that $Ra \not\subset Rb \not\subset R$.
   (d) $a$ is irreducible in $R$ iff (ii) $Ra \neq \{0\}$ and $Ra \neq R$, and (ii) if $b \in R$ is such that $Ra \subset Rb \subset R$, then either $Ra = Rb$ or $Rb = R$.
   Note that “$\not\subset$” means ”proper subset of”.

In the following exercises, let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. This is a subring of the real numbers.

(2) Define $N: R \to \mathbb{Z}$ by $N(a + b\sqrt{2}) := (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$, for $a, b \in \mathbb{Z}$.
   Show that
   (i) $N(1) = 1$,
   (ii) $N(xy) = N(x)N(y)$, and
   (iii) $N(x) = 0$ if and only if $x = 0$.
   (Your proof in (iii) will need to use the fact that $\sqrt{2}$ is irrational.)

(3) Show that $x \in R^\times$ if and only if $N(x) \in \{1, -1\}$.

(4) Show that $R^\times$ is an infinite group. (Hint: find an element of infinite order in $R^\times$.)