Due in class F 26 Oct.

(1) Determine all the normal subgroups of $S_4$. Justify your answer.

(2) For each conjugacy class $C$ in $S_5$, pick an element $g \in C$ and describe the elements of its centralizer group $\text{Cent}_{S_5}(g)$. (Most of them are very easy, using that $\langle g \rangle \leq \text{Cent}(g).$)

(3) For every element $A = \text{Rot}_u(\theta) \in SO(3)$, describe the elements of the centralizer subgroup $\text{Cent}_{SO(3)}(A)$. (Hint: there are basically three distinct cases.)

(4) (Removed.)

(5) Given a set $X$, define a new set $\text{Pair}(X)$ by $\text{Pair}(X) = \{A \subseteq X \mid |A| = 2\}$, the set of subsets of size 2 in $X$. For example, $\text{Pair}\{1, 2, 3, 4\} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

(a) Given an action by a group $G$ on the set $X$, show that $G$ acts on $Y = \text{Pair}(X)$ by the formula

$$\sigma \cdot A := \{\sigma \cdot a \mid a \in A\}.$$

I’m going to call this the “induced action” of $G$ on $\text{Pair}(X)$.

(b) Consider the “standard” action of $S_n$ on $X = \{1, \ldots, n\}$, defined by $\sigma \cdot k := \sigma(k)$. From this we get an “induced” action of $S_n$ on pairs $Y = \text{Pair}(X)$. Describe the orbit(s) of this induced action and compute their size(s), and for any $A = \{a, b\} \in Y$ compute the order of the stabilizer subgroup $\text{Stab}(A) \leq S_n$.

(6) Let $Z = \text{Pair}(Y)$ (the set of “pairs of pairs”). This gets an action by $S_n$ induced by the action on $Y$ of the previous exercise. Describe the orbit(s) of the action and compute their size(s), and for any $S = \{A, B\} \in Z$ compute the order of the stabilizer subgroup $\text{Stab}(A) \leq S_n$.

(Hint: if $S = \{A, B\}$ is an element of $Z$, then the elements $A, B$ of $Y$ must not be equal, but they do not need to be disjoint. It may help to think through an explicit example, e.g., $n = 4$.)

(7) Let $G$ be a finite group, and $H \leq G$ a subgroup.

(a) Consider $\phi: G \to \text{Sym}(G/H)$ defined by $\phi(g)(xH) := gxH$.

Show that $\phi$ is a homomorphism of groups.

(b) Let $K = \text{Ker} \phi$. Show that $K \leq H$.

(c) Show that $[H : K]$ divides $(m - 1)!$, where $m = [G : H]$. (Hint: $\phi(H) \approx H/K$.)

(d) Suppose $[G : H] = p$ is the smallest prime which divides $|G|$. Show that $H = K$ and therefore that $H$ is normal.

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