(1) Let $G$ be a finite group. Write $n = |G|$. Let $p$ be a prime number, and consider the subset $Y := \{ g \in G \mid g^p = e \}$ of elements of $G$ whose order divides $p$.

Let $X = \{ (g_1, \ldots, g_p) \mid g_1g_2 \cdots g_{p-1}g_p = e \}$. This is a subset of the set $G^p$ of ordered $p$-tuples of elements of $G$.

(a) Show that $|X| = n^{p-1}$.

(b) Show that there is a function $\sigma : X \to X$ defined by the rule

$$\sigma(g_1, \ldots, g_p) := (g_p, g_1, \ldots, g_{p-1})$$

and show that this function is a bijection.

(c) Show that $\sigma^p = \text{id}$ in $\text{Sym}(X)$.

(d) Let $X^\sigma = \{ x \in X \mid \sigma(x) = x \}$. Show that $g \mapsto (g, \ldots, g)$ defines a bijection $Y \to X^\sigma$, and thus $|X^\sigma| = |Y|$.

(e) Use Exercise (6) from PS 6 to show that if $p|n$, then $p$ divides $|Y|$.

(f) Conclude that: If $G$ is a finite group whose order is divisible by a prime $p$, then $G$ contains an element of order $p$.

(Remark: I actually did the case of $p = 2$ in class a while ago.)

(2) Fix $n \geq 1$.

(a) Show that $H := \{ \lambda I \mid \lambda \in \mathbb{R}^\times \}$ is a subgroup of $GL_n(\mathbb{R})$ which is isomorphic to $\mathbb{R}^\times$.

(b) Recall that the special linear group $SL_n(\mathbb{R}) = \{ A \in GL_n(\mathbb{R}) \mid \det(A) = 1 \}$ is subgroup of $GL_n(\mathbb{R})$. Show that if $n$ is odd, then there is an isomorphism of groups $GL_n(\mathbb{R}) \approx \mathbb{R}^\times \times SL_n(\mathbb{R})$.

(c) Explain why your proof doesn’t work if $n$ is even.

(3) Let $H$ be a subgroup of $G$.

(a) Show that the rule

$$Hg \mapsto g^{-1}H$$

is a well-defined function, and that it gives bijection $H \setminus G \to G/H$ from the set of right $H$-cosets to the set of left $H$-cosets.

(b) Give an example of a group $G$ and subgroup $H \leq G$, for which the rule

$$Hg \mapsto gH$$

does not give a well-defined function from right $H$-cosets to left $H$-cosets.

(continued...)
(4) Let $G$ be a group, and $N, H \leq G$ two subgroups. Suppose that $N$ is a normal subgroup of $G$.
(a) Show that the subset
\[ NH = \{ nh \mid n \in N, \ h \in H \} \]
is a subgroup of $G$.
(b) Show that $N$ is a normal subgroup of $NH$, and that $N \cap H$ is a normal subgroup of $H$.
(c) Show that there is an isomorphism of groups $H/(N \cap H) \to (NH)/N$.
(d) Give an example of a group $G$ and subgroups $H, K \leq G$ such that $HK = \{ hk \mid h \in H, \ k \in K \}$ is not a subgroup of $G$.

(5) Fix $n \geq 3$ and consider the group $G = \Phi(2^n)$ (=modular units modulo $2^n$ under multiplication). Let $H \subseteq \Phi(2^n)$ be the subset of elements that can be written as $[x]_{2^n}$ for an integer $x$ such that $x \equiv 1 \mod 4$.
(a) Prove that $H$ is a subgroup of $\Phi(2^n)$, and that $|H| = 2^{n-2}$.
(b) Let $a = 5 = 1 + 2^2$. Show that for any $k \geq 0$, we have that
\[ a^{2^k} = 1 + 2^{k+2}y \quad \text{for some odd integer} \ y. \]
(Hint: induction on $k$.)
(c) Show that order([5]_{2^n}) = 2^{n-2} in $G$ (use part (b) to do this). Conclude that $H = \langle [5]_{2^n} \rangle$.
(d) Show that $G \approx H \times K$, where $K = \{ [1]_{2^n}, [-1]_{2^n} \} \leq G$.
This proves that $\Phi(2^n) \approx \mathbb{Z}/2^{n-2} \times \mathbb{Z}/2$ for $n \geq 3$.

Department of Mathematics, University of Illinois, Urbana, IL
E-mail address: rezk@illinois.edu