Due in class F 7 Sep.

(1) In this exercise we construct a field with 4 elements. Assume that we know that $\mathbb{F}_2 = \{0, 1\}$ is a field.

Define $(K, +, \cdot)$ as follows.

- The underlying set of $K$ is $\mathbb{F}_2 \times \mathbb{F}_2 = \{(a, b) \mid a, b \in \mathbb{F}_2\}$, a set with exactly 4 elements: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$.
- $(a, b) + (a', b') := (a + a', b + b')$.
- $(a, b) \cdot (a', b') := (aa' + bb', ab' + ba' + bb')$.

Show that $(K, +, \cdot)$ satisfies the axioms for a field. (*Hint:* To show multiplicative inverses exist, think about $(a, b) \cdot (a + b, b)$.)

(2) A subset $M \subseteq \mathbb{R}$ of the real numbers is well-ordered if every non-empty subset of $M$ contains a smallest element. (The Well-Ordering Principle is just the fact that $\mathbb{N}$ is well ordered.) In each of the following cases, determine whether $M$ is well-ordered.

(a) $M = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.
(b) $M = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.
(c) $M = \{m - \frac{1}{n} \mid m, n \in \mathbb{N}\}$.

(3) Prove the following, where $a, b, c \in \mathbb{Z}$.

(a) If $Zb \subseteq Za$ and $Zc \subseteq Zb$, then $Zc \subseteq Za$.
(b) If $n, m \in Za$ then $xn + yn \in Za$.
(c) We have $Za = Zb$ if and only if $a = \pm b$.

(4) If $S \subseteq \mathbb{Z}$ is a subgroup and $a, b \in S$, then $I(a, b) \subseteq S$.