Due in class W 6 Dec. (#5–8 revised 2 Dec, #1 revised 3 Dec (so \(a \neq 0\)), #7 revised 4 Dec.)

(1) Let \(R\) be an integral domain, and \(a \in R \setminus \{0\}\). Show that \(R/(a)\) is an integral domain if and only if \(a \in R\) is a prime element.

(2) Let \(R\) be a PID. Show that \(R/(a)\) is a field if and only if \(a \in R\) is an irreducible element.

(3) Let \(R\) be a commutative ring, \(p \in R\) an element, and write \(\pi_p: R \rightarrow R/pR\) for the quotient homomorphism. Show that there exists a ring homomorphism \(\varphi_p: R[X] \rightarrow (R/pR)[X]\) such that (i) \(\varphi_p(r) = \pi(r)\) for \(r \in R \subseteq R[X]\), and (ii) \(\varphi_p(X) = X\).

(4) Let \(f = a_nX^n + \cdots + a_1X + a_0 \in \mathbb{Z}[X]\) be a polynomial with integer coefficients. Let \(p \in \mathbb{Z}\) be a prime number such that \(p \nmid a_n\). Show that if \(f = gh\) for some \(g, h \in \mathbb{Z}[X]\) with \(\deg(g), \deg(h) > 0\), then \(\varphi_p(f)\) is a reducible element of \((\mathbb{Z}/p\mathbb{Z})[X]\). Use this to prove that if \(\varphi_p(f)\) is irreducible in \((\mathbb{Z}/p\mathbb{Z})[X]\), then \(f\) is irreducible in \(\mathbb{Q}[X]\).

The following exercises give another proof of the Gauss Lemma. In the following, assume \(R\) is a UFD, and \(\varphi_p\) is defined as in exercise (3).

(5) Show that \(f \in \text{Ker} \varphi_p\) if and only if \(p\) is a common divisor of all the coefficients of the polynomial \(f\).

(6) Show that \(f \in \text{Prim}(R[X])\) if and only if \(\varphi_p(f) \neq 0\) for all irreducible \(p \in R\).

(7) Show that \((R/pR)[X]\) is a domain if \(p\) is irreducible. (Hint: Use the fact that in a UFD, irreducible elements are always prime elements.)

(8) Prove the Gauss Lemma (products of primitives are primitive) using (5) and (6).

(9) Consider an arbitrary symmetric homogeneous polynomial \(f\) of degree 3 in \(F[X_1, X_2, X_3]\), where \(F\) is a field. (i) Show that any such \(f\) can be written

\[
 f = am_1 + bm_2 + cm_3
\]

for some \(a, b, c \in F\), where

\[
 m_1 = X_1^3 + X_2^3 + X_3^3,
 m_2 = X_1^2X_2 + X_1^2X_3 + X_2^2X_1 + X_2^2X_3 + X_3^2X_1 + X_3^2X_2,
 m_3 = X_1X_2X_3.
\]

(ii) Describe how to write \(m_1\), \(m_2\), and \(m_3\) in terms of the elementary symmetric polynomials \(s_1 = X_1 + X_2 + X_3\), \(s_2 = X_1X_2 + X_1X_3 + X_2X_3\), \(s_3 = X_1X_2X_3\).