Due in class M 23 Oct.

Recall that

\[ O(2) = \{ A \in M_{2 \times 2}(\mathbb{R}) \mid AA^\top = I = A^\top A \} \]

is a subgroup of \( \text{GL}_2(\mathbb{R}) \), called the **2nd orthogonal group**. These are exactly the matrices whose columns form an orthonormal basis of \( \mathbb{R}^2 \). Note that if \( A \in O(2) \) then \( \det A \in \{ \pm 1 \} \).

1. Show that if \( A \in O(2) \) with \( \det A = 1 \), then

\[
A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}
\]

for some \( a, b \in \mathbb{R} \) such that \( a^2 + b^2 = 1 \),

while if \( A \in O(2) \) with \( \det A = -1 \), then

\[
A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}
\]

for some \( a, b \in \mathbb{R} \) such that \( a^2 + b^2 = 1 \).

(Hint: what are the unit vectors perpendicular to \((a, b)\)?)

2. Write

\[
R_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad J_\theta := \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.
\]

Verify the identities:

\[
J_{\theta + \pi n} = J_\theta \quad \text{for} \quad n \in \mathbb{Z},
\]

\[
J_\theta = R_{2\theta}J_0,
\]

\[
J_0^2 = I, \quad J_0R_\theta = R_{-\theta}J_0.
\]

Use these to verify that:

\[
J_\alpha J_\beta = R_{2(\alpha - \beta)},
\]

\[
R_\alpha J_\theta = J_{\theta + \frac{1}{2} \alpha},
\]

\[
J_\theta R_\alpha = J_{\theta - \frac{1}{2} \alpha}.
\]

Also, show that every element in \( O(2) \) can be written uniquely as either:

\[ R_\theta \quad \text{for some} \quad \theta \in [0, 2\pi) \quad \text{or} \quad J_\theta \quad \text{for some} \quad \theta \in [0, \pi). \]

(Note: you can also use the identities \( R_{\theta + 2\pi n} = R_\theta \) and \( R_\alpha R_\beta = R_{\alpha + \beta} \), which you don’t need to prove.)

3. Prove that for all \( n \geq 1 \), the subsets

\[
A_n := \{ R_{2\pi(k/n)} \mid k \in \mathbb{Z} \}, \quad B_n := \{ R_{2\pi(k/n)}, J_{\pi(k/n)} \mid k \in \mathbb{Z} \},
\]

are finite subgroups of \( O(2) \). (Note: \( A_n \approx \mathbb{Z}/n \) and \( B_n \approx D_n \) (dihedral group), but I won’t ask you to prove this.)

Date: October 17, 2017.
In the following exercises I write $H := \mathbb{R}^4$, and write the standard basis vectors as

$$1 = (1, 0, 0, 0), \ \hat{i} = (0, 1, 0, 0), \ \hat{j} = (0, 0, 1, 0), \ \hat{k} = (0, 0, 0, 1).$$

(5) Define a binary operation $u, v \mapsto uv: H \times H \to H$ by formula

$$(a, b, c, d)(a', b', c', d') = (aa' - bb' - cc' - dd', \ ab' + ba' + cd' - dc', \ ac' + ca' + db' - bd', \ ad' + da' + bc' - cb').$$

Verify that $(H, +, \cdot)$ is a ring, where $+$ is vector addition and $\cdot$ is the operation I just defined. (This is the ring of quaternions. You don’t need to carefully verify the axioms involving only addition, because these are just standard properties of vector addition. Showing that multiplication is associative is very tedious: be strong!)

(6) Verify the identities:

$$\begin{align*}
\hat{i}^2 &= \hat{j}^2 = \hat{k}^2 = -1 = \hat{i}\hat{j}\hat{k}, \\
\hat{i}\hat{j} &= \hat{k} = -\hat{j}\hat{i}, \\
\hat{j}\hat{k} &= \hat{i} = -\hat{k}\hat{j}, \\
\hat{k}\hat{i} &= \hat{j} = -\hat{i}\hat{k}.
\end{align*}$$

In particular, note that $H$ is not a commutative ring. Also prove that for an element $x \in H$, we have that $xy = yx$ for all $y \in H$ if and only if $x = a1$ for some $a \in \mathbb{R}$.

(7) For $x = (a, b, c, d) \in H$, define $\overline{x} := (a, -b, -c, -d)$, called the conjugate of $x$. Prove that

$$\overline{xy} = \overline{y}\overline{x}, \quad \text{and} \quad x\overline{x} = \overline{x}x = (a^2 + b^2 + c^2 + d^2)1.$$ Use this to show that every non-zero element $x \in H \setminus \{0\}$ has a multiplicative inverse (i.e., an element $y$ such that $xy = 1 = yx$) and give a formula for it.

(8) Let $G = \{ x \in H \mid x1 = 1 \}$, the set of unit length quaternions. Show that $G$ is a group under the operation of multiplication.

(9) A scalar quaternion is an element of the form $\lambda 1$ where $\lambda \in \mathbb{R}$. A vector quaternion is an element of the form $b\hat{i} + c\hat{j} + d\hat{k}$ where $b, c, d \in \mathbb{R}$. Given two vector quaternions $u, v$, write their product $uv$ as a sum $\lambda 1 + w$ of a scalar quaternion and vector quaternion, and express $\lambda$ and $w$ in terms of standard operations on vectors in $\mathbb{R}^3$.

(10) Let $V := \{ b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c \in \mathbb{R} \}$ be the set of vector quaternions. Show that for $x \in G$ (the group of unit length quaternions), the formula

$$R_x(u) := xux^{-1}$$

gives a well-defined $\mathbb{R}$-linear map $V \to V$. Show that, under the obvious identification of $V$ with $\mathbb{R}^3$, the function $R: G \to GL_3(\mathbb{R})$ is a homomorphism of groups. Then show that $\text{Ker} R = \{ \pm 1 \}$. (Hint: use the distributive law of multiplication to check linearity of $R_x$, then show $R_x(\hat{i}), R_x(\hat{j}), R_x(\hat{k}) \in V$.)

(Bonus: Show that $\{ R_x \mid x \in G \} = SO(3)$, so that $SO(3) \approx G/\{ \pm 1 \}$.)