Due in class M 16 Oct.

(1) Determine all the normal subgroups of $S_4$. (Hint: use the same method I used in class for $S_5$.)

(2) Let $G$ act on a set $X$. For $x \in X$ define

$$\text{Stab}(x) := \{ g \in G \mid g \cdot x = x \}.$$ 

Prove that Stab$(x)$ is a subgroup of $G$.

(3) Let $G$ act on a set $X$, and suppose $y = g \cdot x$ for some $x, y \in X$ and $g \in G$. Show that

the function

$$\text{conj}_y: \text{Stab}(x) \to \text{Stab}(y)$$

is well-defined, and gives an isomorphism between the subgroups Stab$(x)$ and Stab$(y)$.

(Thus, elements in the same orbit of the action have *conjugate* stabilizer subgroups, which are therefore isomorphic as groups.)

(4) (Orbit-Stabilizer Theorem.) Let $G$ act on a set $X$, and suppose $x \in X$. Let $H := \text{Stab}(x)$.

(a) Show that the formula

$$\phi(gH) := g \cdot x$$

gives a well-defined function $G/H \to G \cdot x$, from the set of left $H$-cosets to the $G$-orbit containing $x$.

(b) Show that the function is actually a bijection.

(c) Conclude that

$$[G: \text{Stab}(x)] = |G \cdot x|,$$

i.e., the index of the stabilizer subgroup of $x$ is equal to the size of the orbit containing $x$.

In particular, if $G$ is finite, we see that $|G \cdot x|$ must divide $|G|$.

(5) Consider the “tautological” action of $G = S_n$ on $X = \{1, \ldots, n\}$.

(a) Describe all orbit(s) of this action.

(b) Describe the subgroup $H = \text{Stab}(n)$, the stabilizer of the element $n$. Prove that $H \approx S_{n-1}$.

(6) Consider the action of $SO(3)$ on $\mathbb{R}^3$ given by matrix multiplication. As noted in class, the orbits of this action are the sets

$$S_r = \{ x \in \mathbb{R}^3 \mid |x| = r \}, \quad r \geq 0.$$

(a) Show that Stab$(0) = SO(3)$. How does this relate to the orbit-stabilizer theorem?

(b) Let $x = (r, 0, 0)$ for some $r > 0$. Describe the subgroup Stab$(x) \leq SO(3)$, and show that it is isomorphic to $SO(2)$. (Note: you should see that you get the same subgroup for all choices of $r > 0$.)
(c) Show that $\text{Stab}(x) \approx SO(2)$ for every non-zero vector $x$.  

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