Due in class M 25 Sep. Revised to fix problem (4). Revised again to fix problem (10).

1. Let $G \subseteq GL_2(\mathbb{R})$ be the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a \in \{\pm 1\}$ and $b \in \mathbb{Z}$.
   
   (a) Prove that $G$ is a subgroup of $GL_2(\mathbb{R})$.
   
   (b) Let $R := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. For each element of $G$, show how to write it as a product of copies of $R$ and $A$ and/or their inverses. Conclude that $G = \langle R, A \rangle$.
   
   (c) Compute the order of every element of $G$.

2. Let $\phi : G \to H$ be a homomorphism of groups.
   
   (a) Prove that $\phi(e_G) = e_H$, and that $\phi(a^{-1}) = \phi(a)^{-1}$ for all $a \in G$.
   
   (b) Prove that $\phi(a^n) = \phi(a)^n$ for all $n \geq 1$.
   
   (c) Prove that if $K$ is a subgroup of $G$, then the image set $\phi(K) = \{ \phi(k) | k \in K \}$ is a subgroup of $H$.
   
   (d) Prove that if $\phi$ is an isomorphism, then its inverse function is also an isomorphism.

3. Give an example of an isomorphism between $D_3$ and $GL_2(\mathbb{Z}/2)$. (To prove it, it is good enough to compare the two multiplication tables.)

4. Let $G = \mathbb{R} \setminus \{-1\}$, and define a binary operation on $G$ by $x * y := x + y + xy$.
   
   (a) Prove that $(G, *)$ is a group.
   
   (b) Give an isomorphism between $(G, *)$ and $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \cdot)$.

5. Fix $c > 0$ and let $G = (\{-c, c\}, *)$ with group law $x * y = (x + y)/(1 + c^{-2}xy)$. (This group appeared on a previous PS.) Produce an isomorphism of groups
   
   $\phi : G \to (\mathbb{R}, +)$.

   (Feel free to ask for a hint.)

6. Let $n = 2m$ be an even integer $\geq 4$. For each such $n$, give an example of a subgroup of $D_n$ which is isomorphic to $D_m$.

7. Let $G$ be a group with subgroups $A, B \leq G$. Let $H = A \times B$ be the product group of the two subgroups. Let $\phi : H \to G$ be the function $\phi(a, b) := ab$. Suppose:
   
   (a) $ab = ba$ for all $a \in A$ and $b \in B$,
   
   (b) $AB = G$, where $AB := \{ ab | a \in A, b \in B \}$, and
   
   (c) $A \cap B = \{e\}$.
   
   Prove that $\phi$ is an isomorphism of groups.

8. Let $SL_n(\mathbb{R}) = \{ A \in GL_n(\mathbb{R}) | \det(A) = 1 \}$. This is a subgroup of $GL_n(\mathbb{R})$, called the **special linear group**. (In fact, $SL_n(\mathbb{R}) = \ker \det$.)
Show that if \( n \) is odd, then \( GL_n(\mathbb{R}) \approx \mathbb{R}^\times \times SL_n(\mathbb{R}) \). (Hint: use the previous exercise, and the subgroup \( H = \{ \lambda I \mid \lambda \in \mathbb{R}^\times \} \) of \( GL_n(\mathbb{R}) \), which is isomorphic to \( \mathbb{R}^\times \).

Also, explain why your proof doesn’t work when \( n \) is even.

(9) In each of the following cases, give an example of an isomorphism.

(a) \( \Phi(5) \approx \mathbb{Z}/4 \).
(b) \( \Phi(7) \approx \mathbb{Z}/6 \).
(c) \( \Phi(16) \approx \mathbb{Z}/2 \times \mathbb{Z}/4 \).
(d) \( \Phi(24) \approx \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \).

(10) Fix \( n \geq 3 \) and consider the group \( G = \Phi(2^n) \) (=modular units modulo \( 2^n \) under multiplication). Let \( H \subseteq \Phi(2^n) \) be the subset of elements that can be written as \([x]_{2^n}\) for an integer \( x \) such that \( x \equiv 1 \mod 4 \).

(a) Prove that \( H \) is a subgroup of \( \Phi(2^n) \), and that \(|H| = 2^{n-2}\).
(b) Let \( a = 5 = 1 + 4 \). Show that \( a^{2^k} = 1 + 2^{k+2}y \) for some odd integer \( y \).
(c) Show that \( \text{order}( [5]_{2^n} ) = 2^{n-2} \) in \( G \) (use part (b) to do this). Conclude that \( H = \langle [5]_{2^n} \rangle \).
(d) Show that \( G \approx H \times K \), where \( K = \{ [1]_{2^n}, [-1]_{2^n} \} \leq G \). This proves that \( \Phi(2^n) \approx \mathbb{Z}/2^{n-2} \times \mathbb{Z}/2 \).