Due in class M 11 Sep. Revised (5) to make notation consistent.

(1) Given any group $G$, let $a_1, \ldots, a_k$ be a list of elements. Show that if $a_1a_2\cdots a_k = e$, then we must have $a_2\cdots a_k a_1 = e$ and $a_k a_1 \cdots a_k^{-1} = e$.

(2) Given any group $G$, let

$$Z := \{ a \in G \mid ax = xa \text{ for all } x \in G \},$$

the set of elements in $G$ which commute with all other elements of $G$. Show that $Z$ is a subgroup of $G$.

(3) Show that the set $G$ of real $2 \times 2$ matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \neq 0$ in $GL_2(\mathbb{R})$ is a subgroup.

(4) Let $F = \mathbb{Z}/2 = \{[0], [1]\}$, the ring of integers modulo 2. Let $G = GL_2(\mathbb{Z}/2)$, the set of invertible $2 \times 2$-matrices with entries in $\mathbb{Z}/2$, which is a group under multiplication of matrices. List all elements of $G$ (there are six) and compute its multiplication table.

(5) Let $H, K$ be groups, and define a binary operation on $G := H \times K$ by:

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1 h_2, k_1 k_2).$$

(a) Verify that $G = H \times K$ with this operation is a group.

(b) Show that the subsets $H' := \{(h, e_H) \mid h \in H\}$ and $K' := \{(e_K, k) \mid k \in K\}$ are subgroups of $G$, where $e_H \in H$ and $e_K \in K$ are the respective identity elements.

(6) Let $a \in G$ be an element of order 12 in some group. Compute $\text{ord}(a^k)$ for every integer $k$.

(7) Prove that if $\text{ord}(a) = d < \infty$, then $\text{ord}(a^k) | d$ for all integers $k$. Furthermore, show that every positive integer $c$ which divides $d$ is the order of some $a^k$.

(8) Show that if an element $a$ of a group has odd order, then it is the square of a another element of the same order: i.e., $a = b^2$ for some $b$ such that $\text{ord}(b) = \text{ord}(a)$.

(9) This exercise gives a proof of the order theorem for finite abelian groups, different from the one I gave in class. Let $G$ be a finite group with $n$ elements, listed as $G = \{a_1, \ldots, a_n\}$. Show that if $G$ is abelian and $g \in G$, we have that

$$a_1 a_2 \cdots a_n = (a_1 g)(a_2 g) \cdots (a_n g).$$

Conclude from this that $g^n = e$, and thus $\text{ord}(g) | n$.

(10) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, elements of $GL_2(\mathbb{R})$. Show that $A$ and $B$ have finite order, but $AB$ has infinite order. (Thus, the product of elements of finite order need not have finite order.)

(11) Consider the matrices

$$R = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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These are elements of $GL_2(\mathbb{R})$. Consider $G = \langle R, A \rangle$, the subgroup of $GL_2(\mathbb{R})$ generated by $R$ and $A$.

(a) Show that $G$ has exactly six elements. Give a formula for each element in terms of $R$ and $A$.

(b) Determine the orders of all elements in $G$.

(c) Determine which pairs of elements commute in $G$: i.e., for which $x, y \in G$ do we have $xy = yx$?