Due in class M 11 Sep.

(1) (Important.) Let $a, n \in \mathbb{Z}$ with $n \geq 1$. Show that $\gcd(a, n) = 1$ if and only if there exists $b \in \mathbb{Z}$ such that $ab \equiv 1 \mod n$.

(2) Fix a positive real number $c$, and let $S = (-c, c) \subseteq \mathbb{R}$. Consider the formula

$$x \ast y := \frac{x + y}{1 + c^{-2}xy}.$$

Show that this formula gives a well-defined binary operation on $S$, and that this operation makes $(S, \ast)$ into an abelian group. Explain why the formula does not define a group structure on $\mathbb{R}$.

(3) Let $(\mathbb{R}^3, \times)$ be the set of 3d vectors equipped with the operation of vector cross-product. Which axioms of a group does this satisfy?

(4) Consider the set $G$ consisting of the following real matrices:

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Check that $G$ is a subgroup of $GL_2(\mathbb{R})$, and compute its multiplication table.

(5) Determine all the subgroups of the group $G$ of the previous exercise.

(6) Consider the set $G = \{ \pm I, \pm A, \pm B, \pm C \}$, where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

are elements of $M_{2 \times 2}(\mathbb{C})$. Check that $G$ is a subgroup of $GL_2(\mathbb{C})$, and compute its multiplication table.

Department of Mathematics, University of Illinois, Urbana, IL
E-mail address: rezk@illinois.edu