Due in class W 6 Sep.

(1) A subset \( M \subseteq \mathbb{R} \) of the real numbers is well-ordered if every non-empty subset of \( M \) contains a smallest element. (The Well-Ordering Principle is just the fact that \( \mathbb{N} \) is well ordered.) In each of the following cases, determine whether \( M \) is well-ordered.

(a) \( M = [0, 1] = \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \} \).
(b) \( M = \{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \} \).
(c) \( M = \{ m - \frac{1}{n} \mid m, n \in \mathbb{N} \} \).

(2) Prove the following, where \( a, b, c \in \mathbb{Z} \).

(a) If \( \mathbb{Z} b \subseteq \mathbb{Z} a \) and \( \mathbb{Z} c \subseteq \mathbb{Z} b \), then \( \mathbb{Z} c \subseteq \mathbb{Z} a \).
(b) If \( n, m \in \mathbb{Z} a \) then \( x m + y n \in \mathbb{Z} a \).
(c) We have \( \mathbb{Z} a = \mathbb{Z} b \) if and only if \( a = \pm b \).

(3) If \( S \subseteq \mathbb{Z} \) is a subgroup and \( a, b \in S \), then \( I(a, b) \subseteq S \).

(4) Let \( a, b, c \) be integers, with \( \gcd(a, b) = 1 \). Prove that if \( a|bc \) then \( a|c \). Use this to prove that if \( \gcd(a, b) = 1 \) and if \( a|n \) and \( b|n \), then \( ab|n \).

(Use the fact that \( \gcd(a, b) = 1 \) iff \( as + bt = 1 \) for some \( s, t \in \mathbb{Z} \). DO NOT use prime factorization.)

(5) Let \( a_1, \ldots, a_k \) be integers that are pairwise relatively prime, i.e., \( \gcd(a_i, a_j) = 1 \) for all \( i \neq j \). Show that if \( a_i|n \) for all \( i = 1, \ldots, k \), then \( a_1 \cdots a_k|n \). (Hint: use induction on \( k \) and the previous exercise.)

(6) Suppose \( \gcd(a, m) = 1 \) and \( \gcd(a, n) = 1 \). Show that \( \gcd(a, mn) = 1 \). (Ask for hint if you need one.)

(7) Suppose \( m, n \) are odd integers. Show that \( 8|(m^2 - n^2) \).

(8) Suppose \( p \in \mathbb{N} \) with \( p > 1 \) has the property that if \( p|ab \), then \( p|a \) or \( p|b \). Show that \( p \) must be prime. (This is the converse of the statement proved in class.)

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