These problems are about the relationship between vector spaces over $\mathbb{R}$ and vector spaces over $\mathbb{C}$. Note that every vector space over $\mathbb{C}$ is automatically a vector space over $\mathbb{R}$, by “forgetting” about the fact that you can multiply by non-real scalars. This means that if $V$ and $W$ are vector spaces over $\mathbb{C}$, we speak of a map $f : V \to W$ being $\mathbb{C}$-linear or $\mathbb{R}$-linear. Every $\mathbb{C}$-linear map is $\mathbb{R}$-linear, but not conversely.

Recall that if $z = a + bi$ is a complex number, with $a, b \in \mathbb{R}$, then we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$ for the real and imaginary parts of $z$.

Define functions $\text{Re}, \text{Im} : \mathbb{C}^n \times 1 \to \mathbb{C}^n \times 1$ by

$$\text{Re} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \text{Re}(x_1) \\ \vdots \\ \text{Re}(x_n) \end{bmatrix}, \quad \text{Im} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \text{Im}(x_1) \\ \vdots \\ \text{Im}(x_n) \end{bmatrix}.$$ 

(1) Show that $\text{Re}$, $\text{Im}$, and $i \text{Im}$ are $\mathbb{R}$-linear maps, and that they are not $\mathbb{C}$-linear maps.

(2) Show that $\text{Re} \text{Re} = \text{Re}$, that $(i \text{Im})(i \text{Im}) = i \text{Im}$, that $\text{id} = \text{Re} + i \text{Im}$, and that $\text{Re}(i \text{Im}) = 0 = (i \text{Im}) \text{Re}$. Describe the nullspaces and rangespaces of $\text{Re}$, $\text{Im}$, and $i \text{Im}$. (These will be $\mathbb{R}$-subspaces of $\mathbb{C}^{n \times 1}$.

(3) Let $A \in \mathbb{C}^{m \times n}$ be a matrix with complex entries, and write $L_A : \mathbb{C}^n \times 1 \to \mathbb{C}^m \times 1$ for the corresponding $\mathbb{C}$-linear operator defined by $L_A(x) = Ax$. Show that $A \in \mathbb{R}^{m \times n}$ if and only if $L_A \text{Re} = \text{Re} L_A$, if and only if $L_A \text{Im} = \text{Im} L_A$. (Hint: apply the operators to standard basis vectors.)

In the next several problems, we will suppose that $A \in \mathbb{R}^{n \times n}$ is a square real matrix, and that $L_A : \mathbb{C}^n \times 1 \to \mathbb{C}^n \times 1$ is the associated complex-linear map defined by $L_A(x) = Ax$.

(4) Let $\lambda \in \mathbb{R}$. Show that $E_{L_A}(\lambda)$ is invariant (as an $\mathbb{R}$-vector subspace) under both $\text{Re}$ and $\text{Im}$.

(5) Show that if $A \in \mathbb{R}^{n \times n}$ is such that $L_A$ has a complex eigenvector with real eigenvalue $\lambda$, then it has a real eigenvector with the same eigenvalue $\lambda$. (Hint: use (4).)

(6) Give an example of $A \in \mathbb{C}^{2 \times 2}$ which has real eigenvalues, but no real eigenvectors. (Hint: build a diagonalizable matrix by making a suitable choice of eigenvectors and eigenvalues.)
Recall that if \( z = a + bi \in \mathbb{C} \), with \( a, b \in \mathbb{R} \), then its **conjugate** is \( \overline{z} = a - bi \).

Let \( \text{Conj} = \text{Re} - i \text{Im} : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1} \). Note that \( \text{Conj} \) is \( \mathbb{R} \)-linear; it is not \( \mathbb{C} \)-linear.

(7) Show that \( \text{Conj} : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1} \) is a \( \mathbb{C} \)-antilinear involution. (A function \( f : V \to W \) between \( \mathbb{C} \)-vector spaces is **\( \mathbb{C} \)-antilinear** if
\[
 f(c_1 v_1 + c_2 v_2) = c_1 \overline{f(v_1)} + c_2 \overline{f(v_2)}
\]
for all \( v_1, v_2 \in V \) and \( c_1, c_2 \in \mathbb{C} \). A function \( f : V \to V \) is an **involution** if \( ff = \text{id} \).)

(8) Show that if \( T : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1} \) is a \( \mathbb{C} \)-linear map such that \( T \text{Conj} = \text{Conj}T \), then \( \text{Conj} \) takes \( \mathcal{E}_T(\lambda) \) **isomorphically** into \( \mathcal{E}_T(\overline{\lambda}) \) for all \( \lambda \in \mathbb{C} \).

(9) Show that if \( A \in \mathbb{R}^{n \times n} \), then \( L_A \text{Conj} = \text{Conj} L_A \). Conclude that if \( A \in \mathbb{R}^{n \times n} \) and \( \lambda \in \mathbb{C} \) is an eigenvalue of \( A \), then \( \overline{\lambda} \) is also an eigenvalue of \( A \). (Use (8); don’t use determinants here.)

Thus, we have shown that the non-real eigenvalues of real matrices come in conjugate pairs, without using determinants. The standard example is \( A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \), with eigenvalues \( e^{\pm i \theta} = \cos \theta \pm i \sin \theta \).

(10) Given an example of \( A \in \mathbb{C}^{2 \times 2} \) and \( \lambda \in \mathbb{C} \) such that \( \lambda \) is an eigenvalue of \( A \), but \( \overline{\lambda} \) is not an eigenvalue of \( \mathbb{C} \).