PROBLEMS ON FIELDS LIKE $\mathbb{C}$

CHARLES REZK

Pick an element $u \in \mathbb{R}$. Let $K_u$ be the set $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$ of ordered pairs of real numbers. Define binary operations $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$(x_1,y_1) + (x_2,y_2) \overset{\text{def}}{=} (x_1 + x_2, y_1 + y_2),$$

$$(x_1,y_1) \cdot (x_2,y_2) \overset{\text{def}}{=} (x_1x_2 + uy_1y_2, x_1y_2 + x_2y_1).$$

Given a real number $c \in \mathbb{R}$, write $c_K \overset{\text{def}}{=} (c,0) \in K_u$. That is, we can think of $\mathbb{R}$ as "included" in $K_u$, by $c \mapsto c_K = (c,0)$.

(1) Show that for $a,b \in \mathbb{R}$, we have that

$$(a + b)_K = a_K + b_K, \quad (ab)_K = a_K \cdot b_K.$$

(2) Show that for $(x,y) \in K_u$, we have $(x,y) + 0_K = (x,y)$ and that $(x,y) \cdot 1_K = (x,y)$.

(3) Show that the set $K_u$ satisfies all the axioms for a field, except possibly the existence of multiplicative inverses. (That is, it satisfies Halmos’s axioms A1,A2,A3,A4, and B1,B2,B3.)

(4) Show that there exists $\alpha = (x,y) \in K_u$ such that $\alpha \cdot \alpha = u_K$.

(5) Show that if $\alpha = x_K \in K_u$ where $x \in \mathbb{R} \setminus \{0\}$, then there exists a unique element $\beta \in K_u$ such that $\alpha \cdot \beta = 1_K$.

(6) For $\alpha = (x,y) \in K_u$, define $\bar{\alpha} = (x,-y)$. Show that $\alpha \cdot \bar{\alpha} = c_K$, where $c = x^2 - uy^2 \in \mathbb{R}$.

(7) Suppose you are given $u \geq 0$. Describe an element $\alpha \in K_u$ such that $\alpha \neq 0_K$, $\bar{\alpha} \neq 0_K$, and $\alpha \cdot \bar{\alpha} = 0_K$. Explain why this implies that this $K_u$ does not satisfy Halmos’s axiom B4. (Revised: this question and the next originally had signs of $u$ reversed.)

(8) Suppose you are given $u < 0$. Show that $K_u$ satisfies axiom B4, and so is a field.

When $u = -1$, then $K_u$ is the same as $\mathbb{C}$, the field of complex numbers.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL

E-mail address: rezk@math.uiuc.edu

Date: January 24, 2012.