PS 3, SOLUTIONS TO SELECTED PROBLEMS (347, REZK)

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(1) Let $F$ be an ordered field, and suppose $x, y, u, v \in F$. Use the order axioms to show that if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$.

Solution. By hypothesis, $x \leq y$, which (by definition of “$\leq$”) means that either $y - x = 0$ or $y - x \in P$, where $P$ is the positive set of the ordered field $F$. Similarly, the hypothesis $u \leq v$ means (by definition) that either $v - u = 0$ or $v - u \in P$.

We want to show that $x + u \leq y + v$, which by definition of “$\leq$” means that either $(y + v) - (x + u) = 0$ or $(y + v) - (x + u) \in P$. Note that

$$(y + v) - (x + u) = (y - x) + (v - u).$$

We have four cases of our hypotheses.

- Suppose $y - x = 0$ and $v - u = 0$. Then obviously $(y + v) - (x + u) = (y - x) + (v - u) = 0$.
- Suppose $y - x = 0$ and $v - u \in P$. Then $(y + v) - (x + u) = (y - x) + (v - u) = v - u \in P$.
- Suppose $y - x \in P$ and $v - u = 0$. Then $(y + v) - (x + u) = (y - x) + (v - u) = y - x \in P$.
- Suppose $y - x \in P$ and $v - u \in P$. Then $(y + v) - (x + u) = (y - x) + (v - u) \in P$ by axiom (P1) (which says that if $a, b \in P$ then $a + b \in P$).

(2) Let $F$ be an ordered field and suppose $x \in F$. Use the order axioms to show that if $0 < x$, then $0 < x^{-1}$.

Solution. By hypothesis, $x \in P$. Since $x \neq 0$ (by axiom (P3)), there is a reciprocal $x^{-1} \in \mathbb{R}$. To show that $x^{-1} \in P$, note that $x^{-1} \neq 0$ (since otherwise if $x = 0$ then $x \cdot x^{-1} = x \cdot 0 = 0 \neq 1$). Thus, by axiom (P3) (trichotomy), it suffices to show that $-x^{-1} \notin P$.

Suppose $-x^{-1} \in P$; we derive a contradiction. Then $-1 = x \cdot (-x^{-1}) \in P$, but we proved in class that $-1 \notin P$. Thus, we cannot have $-x^{-1} \in P$, as promised.

Note. An alternate proof: observe that $x^{-1} = x \cdot (x^{-1})^2$. We showed in class that squares of non-zero numbers are always positive; therefore (P2) applies to show that $x \cdot (x^{-1})^2$ is positive.

(3) Let $F$ be an ordered field and suppose $x, y \in F$. Use the order axioms to show that if $0 < x < y$ then $0 < y^{-1} < x^{-1}$.

Solution. By hypothesis, $x \in P$ and $y - x \in P$. Therefore $y = (y - x) + x \in P$ by axiom (P1). We have already shown in the previous problem that $x^{-1}, y^{-1} \in P$. It remains to show that $x^{-1} - y^{-1} \in P$.

By algebra,

$$x^{-1}y^{-1}(y - x) = x^{-1} - y^{-1}.$$

We have that $x^{-1}, y^{-1} \in P$, and therefore the product is in $P$ using axiom (P2) (which says that if $a, b \in P$, then $ab \in P$).

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(4) Let \( S = \{ (x, y) \in \mathbb{R}^2 \mid (1-x)(1-y) \geq 1-x-y \} \). Give a simple description of \( S \) involving the signs of \( x \) and \( y \).

Expanding both sides gives the condition \( 1-x-y+xy \geq 1-x-y \). Subtracting to move everything to one side gives

\[
(1-x-y+xy) - (1-x-y) \geq 0,
\]

which simplifies to \( xy \geq 0 \). So \( S = \{ (x, y) \in \mathbb{R}^2 \mid xy \geq 0 \} \), i.e., the union of the first and third quadrants, including the axes.

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