PS 2, SOLUTIONS TO SELECTED PROBLEMS (347, REZK)

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(3) MT 1.29. Let \( x, y, z \) be nonnegative real numbers such that \( y+z \geq 2 \). Prove that \((x+y+z)^2 \geq 4x + 4yz\). When does equality hold?

**Solution.** We have that \( x, y, z \geq 0 \) and \( y+z-2 \geq 0 \). We want to prove that under these hypotheses, \((x+y+z)^2 - (4x + 4yz) \geq 0\).

Expand the left-hand side and regroup:

\[
(x + y + z)^2 - (4x + 4yz) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz - 4x - 4yz
\]

\[
= x^2 + (y^2 - 2yz + z^2) + 2x(y+z-2)
\]

\[
= x^2 + (y-z)^2 + 2x(y+z-2)
\]

This quantity is nonnegative, since the squares \( x^2 \) and \( (y-z)^2 \) are always nonnegative (no matter what \( x, y, z \) are), and our hypothesis says that both \( x \) and \( y+z-2 \) are \( \geq 0 \).

Equality holds (under the hypotheses) if and only if \((x, y, z) = (0, y, y)\). It is straightforward to check that equality holds in this case. Conversely, if \((x+y+z)^2 = 4x + 4yz\), then \((x+y+z)^2 - (4x + 4yz) = 0\). Since \((x+y+z)^2 - (4x + 4yz) = x^2 + (y-z)^2 + 2x(y+z-2)\), and all terms on the right-hand side are non-negative, equality implies \(x^2 = 0\), \((y-z) = 0\), and \(2x(y+z-2) = 0\), i.e., \(x = 0\), and \(y = z\).

**Remark.** It turns out we never used the hypothesis that \( y \) and \( z \) are non-negative here.

The proof I’ve given may seem a little arbitrary, but it is really an example of a useful technique for proving inequalities: write everything in the form \( A \geq 0 \), and look for terms in \( A \) which are clearly non-negative, such as squares or products of known non-negative quantities.

(5) Which of the following functions are well-defined.

(i) \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(n) := n^2 - n \).

(ii) \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(n) := (n^2 + n)/2 \).

(iii) \( f : \mathbb{R} \to [0, +\infty) \) given by \( f(x) := \log(|x|+1) \).

**Solution.** (i) is not well-defined, since \( f(1) = 0 \notin \mathbb{N} \). (ii) is well-defined, since \( n^2 + n = n(n+1) \) is an even positive integer when \( n \) is a positive integer, so \( f(n) \in \mathbb{N} \) for all \( n \in \mathbb{N} \). Likewise, (iii) is well-defined, since \( |x| + 1 \geq 1 \) for all \( x \in \mathbb{R} \), and \( c \geq 1 \) implies \( \log c \geq 0 \).

(6) MT 1.46. Determine the images of the functions \( f : \mathbb{R} \to \mathbb{R} \) defined as follows.

(a) \( f(x) = x^2/(1+x^2) \).

(b) \( f(x) = x/(1+|x|) \).

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Solution. Note that this doesn’t ask for a proof. But I’ll give one anyway. In these cases, drawing a graph will give you a good idea of the answer.

(a) The image is \([0, 1) = \{ b \in \mathbb{R} \mid 0 \leq b < 1 \}\).

Proof. First, note that \(0 \leq x^2 < 1 + x^2\) for all \(x \in \mathbb{R}\), and therefore \(0 \leq x^2/(1 + x^2) < 1\). Thus, the image of \(f\) is subset of \([0, 1)\).

Conversely, suppose \(b \in [0, 1)\). We can produce an \(a \in \mathbb{R}\) such that \(f(a) = b\), simply by solving the equation \(x^2/(1 + x^2) = b\), which gives \(x^2 = b/(1 - b)\). Thus, if \(a = \sqrt{b}/\sqrt{1 - b}\), then \(f(a) = b\), as desired.

(b) The image is \((-1, 1) = \{ b \in \mathbb{R} \mid -1 < b < 1 \}\).

Proof. First, note that

\[-(1 + |x|) < x < 1 + |x|\]

for all \(x \in \mathbb{R}\). You can prove by dividing into cases:

- if \(x \geq 0\), then \(0 \leq x < 1 + x = 1 + |x|\), and
- if \(x \leq 0\), then \(-1 + x < x \leq 0\).

Therefore, the image of \(f\) is a subset of \((-1, 1)\).

Conversely, suppose \(b \in (-1, 1)\). We can produce \(a \in \mathbb{R}\) such that \(f(a) = b\), by finding a solution to the equation \(x/(1 + |x|) = b\). We’ll do this in two cases.

- If \(0 \leq b < 1\), then we have a non-negative solution to \(x/(1 + x) = b\), namely \(a = b/(1 - b)\). Then \(f(a) = b\).
- If \(-1 < b \leq 0\), then we have a non-positive solution to \(x/(1 - x) = b\), namely \(a = b/(1 + b)\). Then \(f(a) = b\).