LECTURE NOTES FOR 347, SPRING 2016

CHARLES REZK

M 14 MAR

Recall the triangle inequality:

\[ |x + y| \leq |x| + |y|. \]

A proof is given in Ch 1 of the text.

Proof. Since \( x \leq |x| \) for all \( x \), we have \( 2xy \leq |2xy| = 2|x||y| \). Add \( x^2 + y^2 \) to both sides to get

\[
x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2 = |x|^2 + 2|x||y| + |y|^2.
\]

This is the same as \( |x + y|^2 = (x + y)^2 \leq (|x| + |y|)^2 \). We showed that if \( a, b \geq 0 \) then \( a^2 \leq b^2 \) implies \( a \leq b \). \( \square \)

The triangle inequality gives a lot of other inequalities. For instance, if you set \( x = b - a \) and \( y = c - b \), then you get

\[
|a - c| \leq |a - b| + |b - c|.
\]

This is obviously saying that the distance from \( a \) to \( c \) is \( \leq \) the sum of distances \( a \)-to-\( b \) and \( b \)-to-\( c \). (You can turn it around, and get the triangle inequality from the new one, by setting \( a = x, c = -y, \) and \( b = 0 \).)

**Proposition 0.1.** If \( a_n \to L \), then \( a_n^2 \to L^2 \).

**Proof.** Idea:

\[
|a_n^2 - L| = |(a_n - L)(a_n + L)| = |a_n - L| \cdot |a_n + L|.
\]

We’ll then show that \( |a_n + L| \) is “eventually bounded from above” by \( M = 2|L| + 1 \), so that \( |a_n^2 - L| \leq M|a_n - L| \). Then we’ll use the fact that \( a_n \to L \) to solve it.

Since \( a_n \to L \), we can choose \( N_1 \) such that \( |a_n - L| < 1 \) for all \( n \geq N_1 \) (i.e., use epsilon = 1). Then

\[
|a_n + L| = |a_n - L + 2L| \leq |a_n - L| + |2L| < 2|L| + 1 = M.
\]

I used the triangle inequality to show that \( |(a_n - L) + 2L| \leq |a_n - L| + |2L| \). Note that \( M \geq 1 > 0 \).

Since \( a_n \to L \), choose \( N_2 \) such that \( |a_n - L| < \epsilon / M \) for all \( n \geq N_2 \) (i.e., use epsilon = \( \epsilon / M \)).

Thus, if \( n \geq N = \max \{ N_1, N_2 \} \) we have

\[
|a_n^2 - L| = |a_n + L| \cdot |a_n - L| < M \cdot (\epsilon / M) = \epsilon.
\]

\( \square \)
Lemma 0.2. If \( k \geq 2 \), \( 1/k^n \to 0 \).

Proof. Step 1. Show that \( n \leq k^n \) by induction on \( n \). Basis step. \( 1 \leq k \). Induction step. If \( n \leq k^n \), then
\[
k^{n+1} = k \cdot k^n > kn = n + (k - 1)n > n + 1
\]
since \( k - 1 \geq 1 \) and \( n \geq 1 \).

Step 2. Therefore \( 1/k^n \leq 1/n \) for all \( n \).

Step 3. Earlier, we showed that if \( |b_n - L| \leq a_n \) and \( a_n \to 0 \), then \( b_n \to L \). Take \( b_n = 1/k^n \), \( L = 0 \), and \( a_n = 1/n \).

\( \square \)

Theorem 0.3. For all positive \( x \in \mathbb{R} \), there exists positive \( y \in \mathbb{R} \) such that \( y^2 = x \).

Proof. Let \( l_n \) be the largest number of the form \( a_n/10^n \), with \( a_n \in \mathbb{N} \), such that \( l_n^2 \leq x \). (This exists by Archimedean property: some multiple of \( 1/10^n \) must be bigger than \( c := \max\{1, x\} \), and \( c^2 \geq x \).)

Every multiple of \( 10^{n-1} \) is also a multiple of \( 10^n \), so \( \langle l \rangle \) is non-decreasing. That is,
\[
l_n = \frac{a_n}{10^n} = \frac{10a_n}{10^{n+1}} \leq \frac{a_{n+1}}{10^{n+1}} = l_{n+1},
\]
because \( 10a_n \leq a_{n+1} \).

Let \( r_n := (a_n + 1)/10^n \). Then \( r_n \) must be the smallest number of the form \( b_n/10^n \) with \( b_n \in \mathbb{N} \) and \( x < r_n^2 \). (This is just because \( a_n/10^n = l_n \), and this is the largest of this form with \( l_n^2 \leq x \).) This implies that \( r_n \) is non-increasing: \( r_{n+1} = \frac{b_{n+1}}{10^{n+1}} \leq \frac{10a_n}{10^{n+1}} = r_n \).

We have that
\[
r_n - l_n = \frac{a_n + 1}{10^n} - \frac{a_n}{10^n} = \frac{1}{10^n}.
\]
Thus \( r_n - l_n = 1/10^n \to 0 \). Last time, we proved that for \( \langle l \rangle \) non-decreasing, \( \langle r \rangle \) non-increasing, with \( r_n - l_n \to 0 \), both \( \langle l \rangle \) and \( \langle r \rangle \) converge to the same limit \( L \).

Since \( l_n \to L \) and \( r_n \to L \), we have \( l_n^2 \to L^2 \) and \( r_n^2 \to L^2 \), by what we proved above. Since \( l_n^2 \leq x \) for all \( n \), we have \( L^2 \leq x \). Similarly, use \( x \leq r_n^2 \) and \( r_n \to L \) to show \( x \leq L^2 \). Therefore \( L^2 \leq x \leq L^2 \), so \( x = L^2 \). That is, \( L \) is a squareroot of \( x \).

\( \square \)

What about nth roots? \( a^x \)? We could try to generalize this argument to show that \( n \)th roots of positive numbers always exist. To do this, we need analogs of facts we proved for squaring:

- If \( a, b \geq 0 \) and \( n \geq 1 \), then \( a < b \) if and only if \( a^n < b^n \).
- If \( a_n \to L \) and \( m \geq 1 \), then \( a_n^m \to L^m \).

I'm not going to do this, but you can if you think about it.

We know that \( n \)th roots are a kind of fractional exponential: \( \sqrt[n]{a} = a^{1/n} \). More generally, we are supposed to have \( a^x \) for \( a, x > 0 \). How do you construct these?

There is a better approach, that you can use once you set up the theory of integration (which I won’t do: see chapter 17.)

- Define the natural logarithm \( \ln x := \int_1^x dt/t \) as a function \( (0, \infty) \to \mathbb{R} \) using an integral.
- Show that \( \ln ab = \ln a + \ln b \) (a simple application of change of variable in the integral: chapter 17.28).
• Show that \(\ln x\) is strictly increasing (easy) and is continuous (a fact which seems to be missing from the textbook: you can use 17.17 to prove it).
• Show that \(\ln: (0, \infty) \to \mathbb{R}\) is actually a bijection, so that it has an inverse function \(\exp: \mathbb{R} \to (0, \infty)\), which must satisfy \(\exp(a + b) = \exp(a) \exp(b)\).
• Define \(a^x := \exp(x \ln a)\).

**Decimal expansion.** (I only talked about this briefly in class.)

Given a infinite decimal, like 2.718281828..., we can produce a monotone sequence:

\[2 \leq 2.7 \leq 2.71 \leq 2.718 \leq \cdots\]

The sequence is bounded, because all the terms are less than 3. This is because of the geometric sum formula:

\[1 + q + q^2 + \cdots + q^k = \frac{q^{k+1} - 1}{q - 1} \quad \text{if } q \neq 0.\]

Thus, for instance

\[
2.71828 = 2 + \frac{7}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{2}{10^4} + \frac{8}{10^5} \\
\leq 2 + \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \frac{9}{10^5} \\
= 2 + \frac{9}{10}(1 + 10^{-1} + 10^{-2} + 10^{-3} + 10^{-4}) \\
= 2 + \frac{9}{10} \cdot \frac{10^{-5} - 1}{10^{-1} - 1} \\
= 2 + \frac{(9)(10^{-5} - 1)}{1 - 10} \\
= 2 + (1 - 10^{-5}) = 3 - 10^{-5} < 3.
\]

A \(q\)-ary sequence (with \(q \geq 2\)) is a sequence of “digits” \(\langle d_n \rangle\), where each \(d_n\) is an integer in \(\{0, 1, \ldots, q - 1\}\). This is something we might like to write as \((0.d_1d_2d_3\ldots)_q\). The associated \(q\)-ary expansion of \(\langle d \rangle\) is the sequence \(\langle l \rangle\), defined by

\[l_n = \sum_{i=1}^{n} \frac{d_i}{q^i} = \frac{d_1}{q} + \frac{d_2}{q^2} + \cdots + \frac{d_n}{q^n}.\]

(Note that \(l_n \in \mathbb{Q}\).) The following proposition should not be controversial.

**Proposition 0.4.** Every \(q\)-ary expansion \(\langle l \rangle\) converges to a real number in \([0,1]\).
Proof. The sequence \( \langle l \rangle \) is clearly monotone non-decreasing. It is bounded below by 0, since all terms are non-negative. It is bounded above by 1, since

\[
\begin{align*}
l_n &= \sum_{i=1}^{n} \frac{d_i}{q^i} \\
&\leq \sum_{i=1}^{n} \frac{q-1}{q^i} \\
&= \frac{q-1}{q} \sum_{i=0}^{n-1} \frac{1}{q^i} \\
&= \frac{q-1}{q} \cdot \frac{q^{-n} - 1}{q^{-1} - 1} = 1 - q^{-n} < 1.
\end{align*}
\]

In particular, this means the sup of \( \{a_n\} \) (which is the limit by monotone convergence) must be between 0 and 1. \( \square \)

Different \( q \)-ary expansions can give the same real number. E.g., 0.499\ldots and 0.5000\ldots.

**Proposition 0.5.** Every \( x \in [0,1) \) has a \( q \)-ary expansion.

**Proof. Sketch.** Construct a \( q \)-ary sequence \((0.d_1d_2\ldots)_q\) just as we did for \( \sqrt{x} \), so that \( l_n := \sum_{i=1}^{n} d_i q^{-i} \) is the largest number of the form \( m/q^n \) which is \( \leq x \). Let \( r_n = l_n + q^{-n} \), and show that \( r_n \) and \( l_n \) converge to the same limit, which must be \( x \). \( \square \)