Proof of strong induction. We prove strong induction using weak induction.

We are told
(a) \( P(1) \) is true.
(b) For all \( k \geq 2 \), if \( P(i) \) is true for all \( i < k \), then \( P(k) \) is true.

Let \( Q(n) \) be the statement "\( P(i) \) is true for all \( i \leq n \)." That is, \( Q(n) = "P(1) \text{ and } P(2) \text{ and } \ldots \text{ and } P(n)"."

Then
(a') \( Q(1) \) is true (by (a)).
(b') If \( Q(n) \) is true, then \( Q(n + 1) \) is true (by (b)).

Thus, weak induction implies to prove \( Q(n) \) for all \( n \). Clearly, \( Q(n) \) implies \( P(n) \).

Square root of 2. Do irrationality of \( \sqrt{2} \) example. Thus, \( P(n) \) is the statement "There is no \( m \in \mathbb{N} \) such that \( \sqrt{2} = m/n \)."

Proof. Show that there is no smallest \( n \) such that \( P(n) \) is true; assuming there is such an \( m \), such that \( \sqrt{2} = m/n \), then note that
\[
1 < 2 < 4, \quad 1 < \sqrt{2} < 2, \quad n < m < 2n, \quad 0 < m - n < n,
\]
then use
\[
\sqrt{2} = \frac{m}{n} = \frac{m(m - n)}{n(m - n)} = \frac{m^2 - mn}{n(m - n)} = \frac{2n^2 - mn}{n(m - n)} = \frac{2n - m}{m - n}.
\]
But by induction, this cannot be true, since \( m - n < n \). \( \square \)

More carefully:
- Basis step. Show \( \sqrt{2} \neq m/1 \) for any integer \( m \). This is equivalent to showing \( 2 \neq m^2 \) for any integer \( m \), which is true since \( m \geq 3 \) implies \( m^2 \geq 9 > 2 \), and \( m = 1, 2 \) don’t work.
- Induction step. Fix \( n \in \mathbb{N} \). We want to show there is no integer \( m \) such that \( \sqrt{2} = m/n \). Suppose there is such an \( m \), and derive a contradiction. If \( \sqrt{2} = m/n \), we know \( m \neq n \), and then by algebra we show that \( \sqrt{2} = (2n - m)/(m - n) \), and therefore \( P(m - n) \) is false. By induction, \( P(m - n) \) is true since \( m - n < n \), so we have derived a contradiction.

The book’s proof is stated in the form of descent, which is a convenient way to phrase things when proving a negative.
Theorem (Descent). Let $Q(n)$ be a sequence of mathematical statements, for $n \in \mathbb{N}$. If (b) below holds, then $Q(n)$ is false for all $n \in \mathbb{N}$.

(b) If $Q(m)$ is true for some $m \in \mathbb{N}$, then there exists an $k \in \mathbb{N}$ with $k < m$ such that $Q(k)$ is true.

This looks weird, because there is nothing that looks like it corresponds to the basis step! In the $\sqrt{2}$ example, $Q(n)$ is “$\sqrt{2} = m/n$ for some integer $m$”.

Proof. Let $P(n)$ be the statement “$Q(n)$ is false”, i.e., $P(n)$ is $\neg Q(n)$. We show:

- $P(1)$ is true. Proof: Suppose $P(1)$ is false, i.e., $Q(1)$ is true. Then (b) says that there exists $k \in \mathbb{N}$ with $k < 1$ such that $Q(k)$ is true. But this is impossible, because there is no $k \in \mathbb{N}$ with $k < 1$.

- For $n \geq 2$, if $P(k)$ for all $k < n$, then $P(n)$. Proof: This is equivalent to: for $n \geq 2$, if $P(n)$ is false, then there exists $k < n$ such that $P(k)$ is false. That is, it is equivalent to: if $Q(n)$ is true, then there exists $k < n$ such that $Q(k)$ is true, which follows from (b).