Polynomials. A real polynomial of one variable is a function \( f: \mathbb{R} \to \mathbb{R} \) of the form \( f(x) = c_k x^k + \cdots + c_0 = \sum_{i=0}^{k} c_i x^i \), for some \( k \geq 0 \). The degree of \( f \) is the largest integer \( d \) such that \( c_d \neq 0 \).

What is degree of \( f(x) = 0 \)? With this definition, it is undefined: there is no largest integer such that \( c_d \neq 0 \) when all \( c_d = 0 \).

A root (or zero) of a polynomial \( f \) is an \( a \in \mathbb{R} \) such that \( f(a) = 0 \).

Here is a standard fact about polynomials.

Lemma 0.1. If \( f \) is a (non-zero) polynomial of degree \( d \), then \( a \in \mathbb{R} \) is a zero of \( f \) if and only if \( f(x) = (x - a) h(x) \) for some polynomial \( h \) of degree \( d - 1 \).

Proof. (Biconditional, so we prove both directions.)

If \( f(x) = (x - a) h(x) \) for some \( h \), then clearly \( f(a) = 0 \).

Conversely, suppose \( f(a) = 0 \). Since \( f \) is non-zero, this implies \( d \geq 1 \). Writing \( f(x) = \sum_{i=0}^{d} c_i x^i = c_0 + \sum_{i=1}^{d} c_i x^i \), we have

\[
    f(x) = f(x) - f(a) = \sum_{i=1}^{d} c_i (x^i - a^i).
\]

We use the identity

\[
    x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) = \sum_{j=1}^{n} x^{n-j}a^{j-1},
\]

which is valid for any \( x, a \in \mathbb{R} \). This has the form

\[
    x^n - a^n = (x - a) h_n(x),
\]

where \( h_n(x) = \sum_{j=1}^{n} x^{n-j}a^{j-1} \) is a polynomial of degree \( n - 1 \) in \( x \).

Putting this in, we get

\[
    f(x) = f(x) - f(a) = \sum_{i=1}^{d} c_i (x^i - a^i) = (x - a) \sum_{i=1}^{d} c_i h_i(x).
\]

The expression \( h(x) = \sum_{i=1}^{d} c_i h_i(x) = c_d x^{d-1} + (\text{lower degree terms}) \) is a polynomial of degree \( d - 1 \). Thus we have shown that if \( a \) is a root, then \( f(x) = (x - a) h(x) \). \( \square \)

Theorem 0.2. Every (non-zero) polynomial of degree \( d \) has at most \( d \) zeroes.
Proof. Induction on \( d \). Let \( f \) be a (non-zero) polynomial of degree \( d \).

**Basis step.** If \( d = 0 \), then \( f(x) = c_0 \neq 0 \), which has no zeroes.

**Induction step.** Let \( d \geq 1 \). If \( f \) has no zeroes, we are done. If \( a \) is a zero of \( f \), we have \( f(x) = (x - a)h(x) \) for some polynomial \( h \) of degree \( d - 1 \), which by induction has at most \( d - 1 \) zeroes. A zero of \( f(x) \) must be either a zero of \( h(x) \), or a zero of \( (x - a) \), which has only one zero \( a \).

Note that it is possible that \( a \) is also a zero of \( h(x) \), in which case \( \#(\text{zeroes of } f) = \#(\text{zeroes of } h) \). If not, then \( \#(\text{zeroes of } f) = \#(\text{zeroes of } h) \).

**Remark 0.3.** In the above statement and proofs, I did not really specify a field. Thus, it is true not only for \( F = \mathbb{R} \), but also for \( F = \mathbb{C} \), and in fact for any other field, such as \( F = \mathbb{F}_2 = \{0, 1\} \).

**Corollary 0.4.** Two real polynomials \( f \) and \( g \) are equal as functions \( \mathbb{R} \to \mathbb{R} \) if and only if their corresponding coefficients are equal.

**Proof.** If the coefficients of \( f \) and \( g \) are equal, then it is obvious they are the same function.

Conversely, suppose \( f(x) = g(x) \) for all \( x \in \mathbb{R} \). Then \( h := f - g \) is the constant function \( h(x) = 0 \) for all \( x \in \mathbb{R} \). However, it is also a polynomial function \( h(x) = \sum a_i x^i - \sum b_i x^i = \sum (a_i - b_i) x^i \). But we showed that if \( h \) is a non-zero polynomial, it has only finitely many roots (in fact, no more than the degree of \( h \)). Since \( h \) has every real number as a root, it must be the zero polynomial (i.e., with coefficients all \( = 0 \)). This implies that \( f \) and \( g \) are the same polynomial.

**Remark 0.5.** The above statement actually holds for any infinite field, such as \( \mathbb{R}, \mathbb{Q}, \) or \( \mathbb{C} \).

It is actually false for any finite field \( F \). For instance, consider the field \( \mathbb{F}_2 = \{0, 1\} \), and the polynomial function \( f(x) = x^2 + x \). Then \( f(0) = 0^2 + 0 = 0 \) and \( f(1) = 1^2 + 1 = 0 \), so \( f \equiv 0 \) as a function \( \mathbb{F}_2 \to \mathbb{F}_2 \).

**Strong induction, pp. 63–68.** Strong induction is a variant of ordinary induction, which is more powerful than ordinary induction.

**Theorem 0.6 (Strong Induction).** Let \( P(n) \) be a sequence of mathematical statements, for \( n \in \mathbb{N} \). If (a) and (b) below hold, then \( P(n) \) is true for all \( n \in \mathbb{N} \).

(a) \( P(1) \) is true.

(b) For \( k \geq 2 \), if \( P(i) \) is true for all \( i < k \), then \( P(k) \) is true.

**Proof.** Let \( Q(n) \) be the statement \( \text{“} P(i) \text{” is true for all } i \in \mathbb{N} \text{ with } 1 \leq i < n \text{”} \).

**Game of Nim.** Game for two players. Start with two equal sized piles of coins. Players take alternate turns; a turn consists of removing any positive number of coins from one of the two piles. The player who removes the last coin wins. (Examples.)

**Proposition 0.7.** Player 2 always has a winning strategy.

**Proof.** Let \( \text{Nim}(n) \) denote the game of Nim with two piles of \( n \) coins each. We claim that the second player always has a winning strategy in \( \text{Nim}(n) \). We prove this using strong induction, so that \( P(n) \) is the statement \( \text{“} \text{Player 2 has a winning strategy in } \text{Nim}(n) \text{”} \).

**Basis step.** In \( \text{Nim}(1) \), player 1 must remove one coin from one of the piles. Then player 2 must remove the remaining coin, and wins.
Induction step. Suppose that player 2 has a winning strategy for Nim(i), for all $i < k$. In Nim(n), player 1 must take a certain number of coins, say $j$ where $1 \leq j \leq k$, from one of the piles. Then player 2 can respond by taking $j$ coins from the other pile. If $j = k$, then player 2 wins; if $j < k$, then what remains is a game of Nim($n - j$) in which player 1 moves first, for which player 2 has a winning strategy, by induction.

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