Geometric sum.

Proposition 0.1. If \( q \in \mathbb{R}, \) and \( q \neq 1, \) then \( \sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1} \) for \( n \in \mathbb{N}. \)

Proof. Basis step. If \( n = 1, \) we have \( \sum_{i=0}^{n-1} q^i = 1 \) and \( \frac{q^n - 1}{q - 1} = 1. \) Induction step. Assume \( \sum_{i=0}^{n-1} q^i = (q^n - 1)/(q - 1), \) we want to show \( \sum_{i=0}^{n} q^i = (q^{n+1} - 1)/(q - 1). \) We have

\[
\sum_{i=0}^{n} q^i = \left( \sum_{i=0}^{n-1} q^i \right) + q^n \\
= \frac{q^n - 1}{q - 1} + q^n \\
= \frac{(q^n - 1) + q^n(q - 1)}{q - 1} = \frac{q^{n+1} - 1}{q - 1}
\]

as desired.

Proposition 0.2. If \( a \in \mathbb{R} \) such that \( 0 \leq a \leq 1, \) and \( n \in \mathbb{N}, \) then \( (1-a)^n \geq 1 - na. \)

Proof. Basis step is clear. Induction step:

\[
(1-a)^{n+1} = (1-a)^n(1-a) \\
\geq (1-na)(1-a) \quad \text{by induction, since } 1-a \geq 0 \\
= 1 - (n+1)a + na^2 \\
\geq 1 - (n+1)a \quad \text{since } na^2 \geq 0.
\]

Note: It seems that we never used the hypothesis that \( 0 \leq a \) here.

Applications of induction, pp. 58–62.

Date: February 12, 2016.
Handshake problem. A handshake party is a party with $n$ married couples, one of which is the host and hostess. The rules are

- spouses do not shake hands with each other,
- the $2n - 1$ people other than the host shake hands of different numbers of people.

The question is: how many hands does the hostess shake?

Give examples of small parties ($n = 2, 3$). Note that the hand shake numbers seem to pair up $(0, 2n - 2), (1, 2n - 2)$, etc. The hostess always ends up shaking $n - 1$ hands.

To prove this, first observe: a person can shake anywhere between 0 and $2n - 2$ hands, which gives $2n - 1$ different possibilities. Since there are $2n - 1$ people other than the host, we see that for every integer $i$ such that $0 \leq i \leq 2n - 2$, there is exactly one person other than the host who shakes exactly $i$ hands. Write $P_i$ for this person, and $H$ for the host, so the party can be written

$$S = \{H, P_0, \ldots, P_{2n-2}\}.$$ 

We will prove that the hostess is $P_{n-1}$ by induction. Thus, let $Q(n)$ be the statement that “in a handshake party of $n$ couples, the hostess shakes $n - 1$ hands”.

**Basis step.** If $n = 1$, then $S = \{H, P_0\}$. In particular, there is only the host and hostess, and no one shakes hands.

**Induction step.** Suppose that the claim holds for a handshake party with $n$ couples (where $n \geq 1$); we want to show the claim for $n$ couples. Let $S = \{H, P_0, \ldots, P_{2n}\}$ be a party with $n + 1$ couples, where $P_i$ labels the person who shakes hands with exactly $i$ people. We want to show that the hostess is $P_{n+1}$.

In the party $S$, the person $P_{2n}$ shakes hands with all other people except one; that other person must be $P_0$, and must be the spouse of $P_{2n}$. Thus $P_0$ and $P_{2n}$ are a couple in this party.

Remove this couple $\{P_0, P_{2n}\}$ from the party. The smaller party consists of $T = \{H, P_1, \ldots, P_{2n-1}\}$, but the labelling is now misleading: the new the handshake numbers all need to be decreased by 1, since all these people shook $P_{2n}$’s hand. Thus, $P_i$ has handshake number $i - 1$ in the new party. The new party is still a handshake party, since the numbers are all decreased by the same amount 1. By induction, the hostess is the person who shakes $n - 1$ hands in the party $T$, i.e., the person we labelled $P_n$. 

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