SETS AND FUNCTIONS, PP. 6–10. I’ll review the basic concepts introduced in Chapter 1 briefly, but I will come back to some of them when I talk about Chapter 2.

A Set is a collection of objects, e.g.,
- set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- set of even integers $\{\ldots, -2, 0, 2, \ldots\}$,
- set of positive integers less than 10, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

There is only one kind of fundamental question you can ask about a set, called membership: given an object $x$ and a set $S$, “is $x$ an element of $S$”? Answer must be yes or no.

Every other question about a set must reduce to questions about membership.

Notation
\[
x \in S \quad \text{means "} x \text{ is an element of } S \text{"}
\]
\[
x \notin S \quad \text{means "} x \text{ is not an element of } S \text{"}
\]

“$\in$” looks sort of like an “E”, indicating “element”.

Examples.
- Setbuilder notation: $S := \{1, 7, 23\}$.
  Means: $1 \in S$, $7 \in S$, $23 \in S$. If $x$ is not one of $1$, $7$, $23$, then $x \notin S$.
- Duplication is irrelevant: $\{1, 2, 3, 1, \sqrt{4}\} = \{1, 2, 3\}$.
- Empty set $\{\} = \emptyset$. “$x \notin \emptyset$” no matter what $x$ is.
- Sets can have sets as members: $S := \{1, 2, \{3, 4, 5\}, \{2\}\}$.
  Here, the set $T = \{3, 4, 5\}$ satisfies $T \in S$.

Standard sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $[n] = \{1, \ldots, n\}$.

**Definition.** Equality of sets. Two sets are equal when they have the same elements. That is, $S = T$ if “$x \in S$” exactly when “$x \in T$”.

**Definition.** Subsets. We say $S$ is a subset of $T$ if every element of $S$ is also an element of $T$. That is, $S \subseteq T$ if “$x \in S$” implies “$x \in T$”.

Puzzle. If $S = \{1, 2, \{3\}\}$, is $2 \in S$? $3 \in S$? $\{3\} \in S$? How about $\{\} \in \{}$? Is $\{3\} \subseteq S$? Is $\{} \subseteq S$?
**Definition.** Specification. If \( A \) is a set, and \( P(x) \) is a mathematical statement about \( x \), then
\[
\{ x \in A \mid P(x) \}
\]
defines a set.

Examples.
- \( S = \{ x \in \mathbb{R} \mid x^2 = 4 \} \). (equal to \( \{ 2, -2 \} \))
- \( S = \{ x \in \mathbb{R} \mid x^2 \leq 4 \} \).
- \( S = \{ x \in \mathbb{R} \mid x^7 - 3x^5 - 2x - 5 = 0 \} \).

**Power set.** Example: power set of \( [3] \).

Example. \( \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \). Intervals.

**Set operations, pp. 9–10.** (These will be covered again, in discussion of logic.)
Union \( S \cup T \), intersection \( S \cap T \), set difference \( S \setminus T \) (or \( S - T \)).
Venn diagrams.
Notion of disjoint sets.
Example. \( A = (A \setminus B) \cup (A \cap B) \), and \( (A \setminus B) \cap (A \cap B) = \emptyset \).
Example. If \( E \) and \( O \) represent sets of even and odd integers, then \( E \cup O = \mathbb{Z} \), and \( E \cap O = \emptyset \).

**Functions, pp. 10–15.** (Key idea is to stress notion of well-definedness.)
A function \( f \) from a set \( A \) to a set \( B \) is a rule which assigns to each \( a \in A \) a single element \( f(a) \in B \). The set \( A \) is called the domain of \( f \), and \( B \) is called the target (or codomain) of \( f \). Write \( f: A \rightarrow B \).

Draw a blob picture.
The image of a function is \( \{ f(a) \in B \mid a \in A \} \), the subset of \( B \) which are actual outputs. (Example: \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 \). Note that there are elements of the target not in the image; this is OK.)

**Important.** The domain and target is part of the information of a function. Examples.
- \( f: \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = x^2 \).
- \( g: \mathbb{R} \rightarrow [0, +\infty) \) given by \( g(x) = x^2 \).
- \( h: [0, +\infty) \rightarrow \mathbb{R} \) given by \( h(x) = x^2 \).

All three functions have the same image, but only \( f \) and \( g \) have the same domain, and only \( f \) and \( h \) have the same target. Thus, \( f, g, h \) are different functions.

**Well-definedness.** When you define a function \( f: A \rightarrow B \), you must be sure that the function you are specifying is well-defined. Typically, you will specify your function by giving a “rule”, that for every \( x \in A \) tells you what \( f(x) \in B \) should be. Being well-defined includes the following.

1. You must be sure that for every \( a \in A \), your rule actually gives an output. For instance,

\[
\begin{align*}
 f: \mathbb{R} \rightarrow \mathbb{R} & \text{ by } f(x) := 1/x \\
\end{align*}
\]

is not well-defined (so not actually a function), because the rule does not make sense when \( x = 0 \).
In this case you can fix this by adding to the rule:

\[
f(x) := \begin{cases} 
1/x & \text{if } x \neq 0, \\
47 & \text{if } x = 0 
\end{cases}
\]

is a well-defined function \( \mathbb{R} \to \mathbb{R} \).

(2) You must be sure that the outputs of your rule are actually elements of the target. For instance,

\[
f : \mathbb{N} \to \mathbb{N} \text{ by } f(n) := n/2
\]

is not well-defined, though it would be well-defined as \( \mathbb{N} \to \mathbb{Q} \).

(3) You must be sure that your rule is unambiguous, and gives a single value as output for each input. For instance, consider \( S := \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \), the set consisting of points on the unit circle. Any element of \( S \) can be written \((\cos \theta, \sin \theta)\) for some \( \theta \in \mathbb{R} \), so you might try to define

\[
f : S \to \mathbb{R} \text{ by } f(\cos \theta, \sin \theta) := \theta.
\]

But this is not well-defined, since \((\cos \theta, \sin \theta) = (\cos(\theta + 2\pi n), \sin(\theta + 2\pi n))\) for any \( n \in \mathbb{Z} \). There are multiple choices of \( \theta \) for each point, and they give different outputs.

You can fix this by observing that for \((x, y) \in S\) there is a unique \( \theta \in [0, 2\pi) \) such that \((x, y) = (\cos \theta, \sin \theta)\), and using this fact to define your function. Thus, you end up with \( f : S \to \mathbb{R} \) whose image is \([0, 2\pi)\).

Why are we so careful about domain and target? We might want to make a statement about all functions \( A \to B \) of a certain type.

**Theorem** (Maximum principle). Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then \( f \) attains its maximum value; that is, there exists \( c \in [a, b] \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \).

The point is that this is completely false for functions on other kinds of domains, for instance \( f : [-1, 1] \setminus \{0\} \to \mathbb{R} \) defined by \( f(x) := 1/x \).

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