Introduction. Discuss the format of the course, from the course plan.

Proofs and the goal of the course. This course centers on the idea of mathematical proof. Starting now, proofs are not something incidental to the ideas of the course, but rather are essential. The goal is that after this course, you should be able to

- read and write proofs of mathematical statements,
- be able to determine whether a given argument is a rigorous proof.

Example: even and odd integers.

Claim. The sum of two odd integers is an even integer.

This is a mathematical statement, because it has a truth value; it is either true or false. This one is true. To demonstrate this, we want to give a proof.

The following is not a proof.

Not a proof. 3 + 5 = 8. □

We need an acceptable notion of what it means to be even or odd. You can probably come up with a bunch of possibilities (e.g., an integer is even if its decimal expansion ends in 0, 2, 4, 6, 8). That is fine. But in order to proceed, we need to fix a definition once and for all.

Definition. An integer $n$ is said to be even if $n = 2k$ for some integer $k$.

An integer $n$ is said to be odd if $n = 2k + 1$ for some integer $k$.

Proof of Claim. Suppose that $m$ and $n$ are odd integers. Then, by definition of “odd integer”, there exist integers $k$ and $\ell$ such that $m = 2k + 1$ and $n = 2\ell + 1$. Therefore

$$m + n = (2k + 1) + (2\ell + 1) = 2(k + \ell + 1),$$

which is an even integer, since it is two times the integer $k + \ell + 1$. □

Remarks.

(1) We used the definitions of the terms which appeared in the claim.
(2) The proof started with hypotheses, and from these we deduced the conclusion.
(3) The argument is expressed as a little essay, with complete sentences.
(4) I introduced my characters: when I introduce the symbols $m, n, k, \ell$, I carefully say what sort of objects they are.
Note that although I carefully defined “even” and “odd”, there are other notions which are not defined: such as “integer”, and “+”.

There are also some possible criticisms:

(5) I used some facts about integers without explanation. What facts did I use?
Here’s another one.

Claim. The square of an even integer is even. The square of an odd integer is odd.

Proof. This is really two statements.

• Suppose \( n \) is even. Then \( n = 2k \) for some integer \( k \). Then \( n^2 = 4k^2 = 2(2k^2) \) is twice an integer, so is even.

• Suppose \( n \) is odd. Then \( n = 2k + 1 \) for some integer \( k \). Then \( n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \) is odd.

\[ \square \]

Claim. For any odd integer \( n \), the number \((n - 1)/2\) is an even integer.

This is also a mathematical statement. This one is false.

Counterexample. Suppose that \( n = 3 \). Then \((n - 1)/2 = (3 - 1)/2 = 1\), which is not an even integer, since it is not twice an integer.

The counterexample is actually a proof that the claim is false. The claim said “For any odd integer \( n \ldots \)”, so to prove it is wrong, it is enough to give a single value of \( n \) satisfying the hypothesis (that \( n \) is odd) for which the conclusion \((n - 1)/2\) is an even integer) is false.

Example: perfect squares.

Proposition. Every odd integer is a difference of perfect squares.

A perfect square is just an integer of the form \( k^2 \), where \( k \) is itself an integer. The proposition asserts that if \( n \) is an odd integer, then you can find integers \( a \) and \( b \) such that \( n = a^2 - b^2 \). The proof easy, because it turns out that something even nicer is true.

Proposition. Suppose \( n \) is an odd integer, so that \( n = 2k + 1 \) for some integer \( k \). Then

\[ (k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n. \]

This is a very short proof, but completely adequate.

Here’s a question:

• What if \( n \) is even? When can \( n \) be a difference of perfect squares?

Discuss this. Conclude that the right answer is

Proposition. An even integer is a difference of perfect squares if and only if it is a multiple of 4.

The phrase “if and only if” here means that, for an even integer \( n \),

• if \( n \) is a difference of two perfect squares, then \( n \) is a multiple of 4, and

• if \( n \) is a multiple of 4, then it is a difference of two perfect squares.
When you prove this, you should prove these two parts separately.

Proof is left for homework.

A mathematical statement can be more complicated.

**Conjecture** (Goldbach). *For every even integer \( n \) such that \( n \geq 4 \), there exist prime integers \( p \) and \( q \) such that \( p + q = n \).*

For instance, \( 4 = 2 + 2 \), \( 6 = 3 + 3 \), \( 8 = 3 + 5 \), \( 10 = 3 + 7 \) (or \( 5 + 5 \), etc.). This statement involves another concept which we know, but which I haven’t defined yet: “prime”. Note the phrases “for every” and “there exist”; these are quantifiers. It is a mathematical statement, but one whose answer isn’t known!