Dear All;

On Thursday 11/12, a student came to my office hours asking about clarification for problems that aren’t "Chain Rule-y": this means substitution problems where one cannot visualize the function to be integrated as the outcome of an application of the chain rule in a single step. We did one example like this in lecture. Here is one more:

Find the general antiderivative:

$$\int x^3(x^2 + 1)^{3/2} \, dx$$

We see that $x^2$ looks like the derivative of $x^3$, but usually we choose our $u$ based on the function that we see the derivative of. Substituting $u = x^3$ would lead us to a $du$ inside the function that is raised to the 3/2 power. We only want $du$ to appear outside of a function. Hence, this does not look "Chain Rule-y".

Instead, try letting $u = x^2 + 1$: Then $du = 2x \, dx$ and $\frac{1}{2} \, du = x \, dx$. However, this still does not look like our problem: we see an $x^3$ and a $dx$, but not an $x \, dx$. Let’s rewrite the original problem without changing the function to be integrated, using the fact that $x^3 = x^2 \cdot x$:

$$\int x^3(x^2 + 1)^{3/2} \, dx = \int x^2(x^2 + 1)^{3/2} x \, dx$$

Aha! Now I see something that looks like part of my $u$ choice: the $x^2$ before the $(x^2 + 1)^{3/2}$. Also, I see my $x \, dx$.

We can solve for $x^2$ from our choice of $u$:

$$u = x^2 + 1 \iff x^2 = u - 1$$

Then, we perform the substitution...

$$\int x^2(x^2 + 1)^{3/2} x \, dx \Rightarrow \int (u - 1)u^{3/2}\left(\frac{1}{2} \, du\right) = \frac{1}{2} \int u^{5/2} - u^{3/2} \, du =$$

$$\frac{1}{2} \left( \frac{2}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right) + C = \frac{1}{7} u^{7/2} - \frac{1}{5} u^{5/2} + C$$

Then, we must return to our expression in $x$ since this is an indefinite integral:

$$\frac{1}{7} (x^2 + 1)^{7/2} - \frac{1}{5} (x^2 + 1)^{5/2} + C$$

If you’d like to try more problems like this, please see Problems 46-48 in your textbook. Caveat: once you find your antiderivative and take the derivative to check your work, the algebraic manipulations required to show that you have the same function you started with can be challenging!