Example: Find the equation of the tangent line to the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) at the point where \( x = 1 \) and \( y > 0 \).

Solution:

To solve this problem, we need:

(a) The \( y \)-value on the ellipse where \( x = 1 \) and \( y > 0 \),
(b) An expression for \( \frac{dy}{dx} \), and the value of this expression at the point we found from part (a):
(c) The point-slope form of an equation to use our information from (a) and (b) to find the tangent line.

First we deal with part (a): Plug in \( x = 1 \) to the equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) to obtain an equation in \( y \) and solve:

\[
\frac{1}{4} + \frac{y^2}{9} = 1 \iff \frac{y^2}{9} = 1 - \frac{1}{4} \iff y^2 = \frac{27}{4} \iff y = \pm \frac{\sqrt{27}}{2} = \pm \frac{3\sqrt{3}}{2}
\]

We choose the positive value \( y = \frac{3\sqrt{3}}{2} \) as requested.

Now we deal with part (b): We will use Implicit Differentiation:

\[
\frac{d}{dx} \left( \frac{x^2}{4} + \frac{y^2}{9} \right) = \frac{d}{dx} (1) \Rightarrow \frac{1}{4} (2x) + \frac{1}{9} (2y) \frac{dy}{dx} = 0 \Rightarrow
\]

\[
y \frac{dy}{dx} = -\frac{1}{2} \left( \frac{9}{2} \right) x \Rightarrow \frac{dy}{dx} = -\frac{9}{4} \left( \frac{x}{y} \right)
\]

We need the value of \( \frac{dy}{dx} \) at the point \( \left( 1, \frac{3\sqrt{3}}{2} \right) \):

\[
-\frac{9}{4} \left( \frac{1}{\frac{3\sqrt{3}}{2}} \right) = -\frac{9}{4} \left( \frac{2}{3\sqrt{3}} \right) = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}
\]

Now we can do Part (c) and finish the problem: using the point-slope formula we have

\[
y - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} (x - 1)
\]
Which is the equation to the tangent line to the ellipse at the point \( \left( 1, \frac{3\sqrt{3}}{2} \right) \). Pictured below is a graph of the ellipse and the tangent line.
(2) Correction to first 3.6 Example:

In the first example from Section 3.6, we found that

\[
\frac{d}{dx} \ln(\cot(x)) = -\frac{\csc^2(x)}{\cot(x)}
\]

Which is TRUE. However, I claimed it simplified to \(-\frac{\csc(x)}{\sec(x)}\) which it does not. Instead the simplification is \(-\csc(x)\sec(x)\). This was the result of my trusting a solutions manual for another book, and I apologize for any confusion it may have caused. **The main point is that when you take the derivative of a function involving trig functions, there may be more than one equivalent way to write the answer!** A helpful student checked this and alerted me to it! Thank you.