Consider the Markov chain with state space \( \{0, 1, 2, \ldots, N\} \) where we choose
\[
\begin{align*}
p_{i,i+1} &= \frac{1}{4}, \quad i = 0, 1, \ldots, N-1, \\
p_{i,i-1} &= \frac{3}{4}, \quad i = 1, 2, \ldots, N, \\
p_{0,0} &= \frac{3}{4}, \quad p_{N,N} = \frac{1}{4}.
\end{align*}
\]
Then

1. Show that this Markov chain is irreducible;
2. Show that this Markov chain is reversible;
3. Compute the invariant distribution \( \pi \) of this Markov chain;
4. Compute \( \pi_0 \) asymptotically as \( N \to \infty \).

**Solution.**

1. Choose \( i < j \). There is a path from \( i \) to \( j \) of length \( j - i \) and each step on this path has probability \( 1/4 \), so we have
\[
p_{i,j}(j - i) = \frac{1}{4^{j-i}} > 0.
\]
Similarly, if \( i > j \), then
\[
p_{i,j}(i - j) = \frac{1}{4^{i-j}} > 0.
\]
2. Let us choose any vector \( \lambda \) where \( \lambda_k = \lambda_0 3^{-k} \). Then
\[
\begin{align*}
\lambda_i p_{i,i+1} &= \lambda_0 3^{-i} \frac{1}{4}, \\
\lambda_{i+1} p_{i+1,i} &= \lambda_0 3^{-(i+1)} \frac{3}{4},
\end{align*}
\]
and these are equal. Similarly,
\[
\begin{align*}
\lambda_i p_{i,i-1} &= \lambda_0 3^{-i} \frac{3}{4}, \\
\lambda_{i-1} p_{i-1,i} &= \lambda_0 3^{-(i-1)} \frac{1}{4},
\end{align*}
\]
and these are equal.
3. Since the $\lambda$ mentioned above is in detailed balance with the Markov chain, we can obtain a stationary distribution by rescaling, so we have $\pi$ with

$$\pi_i = \frac{\lambda_i}{\sum_{k=0}^{N} \lambda_k} = \frac{3^{-i} \lambda_0}{\sum_{k=0}^{N} \lambda_0 3^{-k}} = \frac{3^{-i}}{\sum_{k=0}^{N} 3^{-k}}$$

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4. Finally, as $N \to \infty$, we have

$$\pi_0 \to \frac{1}{\sum_{k=0}^{\infty} 3^{-k}} = \frac{1}{1-1/3} = \frac{3}{2} = 2/3.$$