1. (Problem 3.1.1 from Keener.) Verify that the solution of
\[ u''(x) = f(x), \quad u(0) = 0, \quad u(1) = 0, \]
is given by
\[ u(x) = \int_0^1 k(x, y) f(y) \, dy, \]
where
\[ k(x, y) = \begin{cases} 
  y(x - 1), & 0 \leq y < x \leq 1, \\
  x(y - 1), & 0 \leq x < y \leq 1.
\end{cases} \]

2. (Problem 3.2.1 from Keener.) Consider the functional \( T: L^2[0, 1] \to \mathbb{R} \) by \( T(f) = f(0) \). Prove that this is a linear functional, but show that it is unbounded on \( L^2 \) (using the standard definition of \( L^2 \) norm).

However, now consider the functional \( T: C^0[0, 1] \to \mathbb{R} \) defined in the same way, i.e. \( T(f) = f(0) \), and use the “uniform norm” on \( C^0 \), i.e.
\[ \|f\| = \sup_{x \in [0,1]} f(x). \]
Show that \( T \) is now a bounded linear functional.

(However, note that the Riesz theorem doesn’t do anything for us here, since \( C^0 \) is not a Hilbert space — this norm does not come from an inner product.)

3. Let \( A \) be an \( n \times n \) matrix, and consider the linear transformation \( A: \mathbb{R}^n \to \mathbb{R}^n \).

(a) First assume that \( A \) is self-adjoint. Show that we can reorder the eigenvalues of \( A \) so that
\[ |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|, \]
and show
\[ \|A\| = |\lambda_1|. \]

(b) Now let \( A \) be general. Show that \( A^*A \) is self-adjoint, and moreover that the eigenvalues of \( A^*A \) are non-negative. (Then from (a), \( \|A^*A\| \) is the largest eigenvalue of \( A^*A \).

Hint: Let \( x \) be an eigenvector of \( A^*A \), consider the inner product
\[ \langle A^*Ax, x \rangle. \]

(c) Prove that \( \|A\| = \sqrt{\|A^*A\|}. \)

Hint: The best way to do this is in two steps. First show \( \|A^2\| = \|A\|^2 \) (think about the definition of the operator norm). Then show \( \|A^*A\| = \|A^2\|. \)

NB. Thus to compute \( \|A\| \), we simply have to compute the square root of the largest eigenvalue of \( A^*A \). (In fact, the eigenvalues of \( A^*A \) are typically called the singular values of \( A \) for this and other reasons; we will see these again.)
4. (Problem 3.4.4 from Keener.) Find the adjoint kernel for an integral operator when we define the inner product to be

\[ \langle f, g \rangle = \int_a^b f(x)g(x)\omega(x) \, dx. \]

More specifically, what we mean here is that if we assume that \( K \) is the integral operator

\[ (Ku)(x) = \int_a^b k(x, y)u(y) \, dy, \]

then the adjoint of \( K \) (under the inner product above!) should have the representation

\[ (K^*u)(x) = \int_a^b k^*(x, y)u(y) \, dy \]

for some function \( k^*(x, y) \). Determine \( k^* \). (To make things simpler, assume everything is real-valued.) Work out explicitly what the formula for \( K^*K \) is, and compute \( \|K^*K\| \).

5. Consider the differential equation

\[ x' = ax, \quad x(0) = x_0. \]

We know that the solution to this differential equation is \( x(t) = x_0e^{at} \), but let us prove this using Neumann iterates. Prove that the differential equation is equivalent to the integral equation

\[ x(t) = x_0 + \int_0^t ax(s) \, ds, \]

and then solve this integral equation iteratively.

6. (Problem 3.6.6 from Keener.)

(a) Show that the solution of

\[ u(x) = 1 + \int_0^x s \ln(s/x)u(s) \, ds \]

satisfies the differential equation

\[ u'' + x^{-1}u' + u = 0, \quad u(0) = 1, \quad u'(0) = 0. \]

(b) Use Neumann iterates to show that

\[ u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{2k}}{(k!)^2}. \]

(This is actually the zeroth Bessel function.)