Methods of Mathematical Physics - 556 X1
Homework 3 Solutions

1. (Problem 2.1.1 from Keener.) Verify that $\ell^2$ is an inner product space. Specifically, show that if $x, y \in \ell^2$, then

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$$

is defined and satisfies the properties of an inner product. (Here we’re assuming that our sequences are real, so no need for the complex conjugate.)

Hint: Think about how we proved Bessel’s Inequality in class.

Solution. The hardest part of this will be to show that if $x, y \in \ell^2$, then $\langle x, y \rangle$ is finite and $x + y \in \ell^2$; verifying everything else will be straightforward. We have

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n \to \infty} \sum_{k=1}^{n} x_k y_k.$$ 

By the Cauchy-Schwarz inequality for the standard dot product on $\mathbb{R}^n$, we know that

$$\left( \sum_{k=1}^{n} |x_k y_k| \right)^2 \leq \sum_{k=1}^{n} x_k^2 \cdot \sum_{k=1}^{n} y_k^2 \leq \sum_{k=1}^{\infty} x_k^2 \cdot \sum_{k=1}^{\infty} y_k^2.$$ 

The right-hand side is finite, and moreover it is independent of $n$, and thus $\sum_{k=1}^{\infty} |x_k y_k|$ is a convergent sequence. Since we have taken absolute values, this means that the series $\sum_{k=1}^{\infty} x_k y_k$ is an absolutely convergent sequence, and thus converges as well. Moreover, note now that if $x, y \in \ell^2$, then

$$\sum_{k=1}^{\infty} (x_k + y_k)^2 = \sum_{k=1}^{\infty} x_k^2 + 2x_k y_k + y_k^2 < \infty,$$

so $x + y \in \ell^2$. Of course $\alpha x \in \ell^2$ for all $\alpha \in \mathbb{R}$ as well, and this makes $\ell^2$ a vector space. Proving that the remaining axioms of an inner product are satisfied is straightforward at this point.

2. (Problem 2.1.3 from Keener.) Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is a Cauchy sequence. Since the reals are complete, this means it converges. To which number does this sequence converge?

Solution. We need to check that for any $\epsilon > 0$, there is an $N$ such that $n, m > N$ means that

$$|x_n - x_m| < \epsilon.$$ 

But we have

$$|x_n - x_m| = \left| \sum_{k=m+1}^{n} \frac{1}{k} \right| \leq \sum_{k=m+1}^{n} \frac{1}{k^2}.$$ 

But we can replace a sum with an integral with only adding perhaps a constant (think of the Riemann sums, for example), and we have

$$\left| \sum_{k=m+1}^{n} \frac{1}{k^2} \right| \leq \int_{m}^{n} \frac{1}{x^2} dx = \frac{1}{m} - \frac{1}{n} = \frac{n - m}{nm} < \frac{1}{m}.$$ 

So if we choose $N = 1/\epsilon$, then if $m, n > N$ then $|x_n - x_m| < \epsilon$. Therefore this is a Cauchy sequence and thus converges. (Of course, replacing the factorial with a square threw away a lot — this sum converges much faster than $k^{-2}$.)
Finally, we know from calculus that
\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \]
so
\[ \sum_{k=1}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} - 1 = e - 1. \]

3. Show that the sequence \( x_n = \sum_{k=1}^{n} \frac{1}{k} \) is not a Cauchy sequence.

Solution. Let us try to bound \(|x_n - x_m|\), where we get
\[ |x_n - x_m| = \left| \sum_{k=m+1}^{n} \frac{1}{k} \right| \geq \int_{m+1}^{n} \frac{1}{x} \, dx = \log \left( \frac{n}{m+1} \right). \]
Choose \( n = \alpha (m + 1) \), then this difference is at least \( \log \alpha \). This means that no matter how large \( m \) is, we can make this difference large by choose \( n \) large enough. Therefore the sequence is not Cauchy.

4. Consider the Hilbert space \( L^2 \). Prove that the list of vectors
\[ \{ \cos(nx) \}_{n=0}^{\infty} \cup \{ \sin(nx) \}_{n=1}^{\infty} \]
is an infinite orthogonal list in \( L^2 \) with respect to the inner product
\[ (f, g) = \int_{0}^{2\pi} f(x) g(x) \, dx. \]

Hint: You will need some trigonometric identities to solve this problem, e.g. you will need to compute integrals like
\[ \int_{0}^{2\pi} \cos(mx) \cos(nx) \, dx. \]
Recall that we can use Euler’s formula to get, for example,
\[ \cos((m + n)x) = \text{Re}(e^{i(m+n)x}) = \text{Re}(e^{imx}e^{inx}) = \text{Re}((\cos(mx) + i \sin(mx))(\cos(nx) + i \sin(nx))) = \cos(mx) \cos(nx) - \sin(mx) \sin(nx). \]
If you recombine these formulas in a clever way, you can do all of the integrals.

Solution. We need to show that
\[ \int_{0}^{2\pi} \cos(mx) \cos(nx) \, dx = \delta_{mn} C_m, \]
\[ \int_{0}^{2\pi} \sin(mx) \sin(nx) \, dx = \delta_{mn} S_m, \]
\[ \int_{0}^{2\pi} \cos(mx) \sin(nx) \, dx = 0, \]
where \( C_m \) and \( S_m \) are some constants, and then we are done.

Use the formula above for \( \cos((m+n)x) \), and note then that
\[ \cos((m-n)x) = \cos(mx) \cos(nx) + \sin(mx) \sin(nx) \]
(cosine is even and sine is odd!) and then we have
\[ \cos((m+n)x) + \cos((m-n)x) = 2 \cos(mx) \cos(nx). \]
Therefore
\[ \int_{0}^{2\pi} \cos(nx) \cos(mx) \, dx = \frac{1}{2} \int_{0}^{2\pi} \cos((m+n)x) + \cos((m-n)x) \, dx = \frac{1}{2}(2\pi\delta_{m+n,0} + 2\pi\delta_{m-n,0}). \]

Since \( m, n \geq 0 \), we can only have \( m + n = 0 \) if \( m = n = 0 \). Then we have
\[ \int_{0}^{2\pi} \cos(nx) \cos(mx) \, dx = \begin{cases} \frac{2\pi}{2}, & m = n = 0, \\ \pi, & m = n \neq 0, \\ 0, & m \neq n. \end{cases} \]

Similarly, we have
\[ \sin(mx) \sin(nx) = \frac{1}{2}(\cos((m-n)x) - \cos((m+n)x)), \]
and thus
\[ \int_{0}^{2\pi} \sin(nx) \sin(mx) \, dx = \frac{1}{2} \int_{0}^{2\pi} \cos((m-n)x) - \cos((m+n)x) \, dx = \frac{1}{2}(2\pi\delta_{m-n,0} - 2\pi\delta_{m+n,0}). \]

Since \( m, n > 0 \), we cannot have \( m + n = 0 \). Then
\[ \int_{0}^{2\pi} \sin(nx) \sin(mx) \, dx = \begin{cases} \pi, & m = n, \\ 0, & m \neq n. \end{cases} \]

Finally, we need the other Euler’s formula, namely
\[ \sin((m+n)x) = \text{Im}(e^{i(m+n)x}) = \text{Im}(e^{inx}e^{inx}) = \text{Im}((\cos(mx) + i \sin(mx))(\cos(nx) + i \sin(nx))) = \sin(mx)\cos(nx) + \cos(mx)\sin(nx). \]

This gives
\[ \sin((m+n)x) + \sin((m-n)x) = 2\sin(mx)\cos(nx), \]
so
\[ \int_{0}^{2\pi} \sin(mx) \cos(nx) \, dx = \frac{1}{2} \int_{0}^{2\pi} \sin((m+n)x) + \sin((m-n)x) \, dx = 0 \]
(recall that \( \sin(0x) = 0 \) for all \( x \) and thus its integral is 0 as well).

5. (Problem 2.2.1 from Keener.) Find the best quadratic polynomial fit to the function \( f(x) = |x| \), where we choose as inner product
\[ \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x) \, dx, \]
for each of the weights \( \omega(x) = 1, \sqrt{1-x^2}, (1-x^2)^{-1/2} \).

Hint: You might find it convenient to compute some orthogonal polynomials for each weight and then compute the answer in terms of these polynomials — and we already did most of the work here on the last homework!

Solution. See attached Mathematica notebook/pdf.

6. (Problem 2.2.9 from Keener.) Suppose that \( \{\phi_n(x)\}_{n=0}^{\infty} \) is a set of orthonormal polynomials, where we choose the inner product
\[ \langle f, g \rangle = \int_{a}^{b} f(x)g(x)\omega(x) \, dx, \]
\( (\omega(x) > 0) \) and assume that \( \phi_n(x) \) is a polynomial of degree \( n \) with leading coefficient \( k_n \) (specifically, we mean that
\[ \phi_n(x) = k_n x^n + \text{terms of power} \ n - 1 \text{ or less}. \]

Then show:
(a) If $f$ is a polynomial of degree less than $n$, then $\langle \phi_n, f \rangle = 0$.

(b) Show that every polynomial of degree $n$ can be written in the form

$$\sum_{i=0}^{n} \alpha_i \phi_i$$

for some numbers $\alpha_i$.

(c) the polynomials satisfy a recurrence relation of the form

$$\phi_{n+1}(x) = (A_n x + B_n) \phi_n(x) - C_n \phi_{n-1}(x),$$

for every $n$, where $A_n = k_{n+1}/k_n$. Compute $B_n, C_n$ in terms of $A_n, A_{n-1}, \phi_n$.

**Hint:** What do we know about $\phi_{n+1}(x) - A_n x \phi_n(x)$? Use part (b), take the inner product with $\phi_j$, what do you get? Also, notice that for this inner product, $\langle xf, g \rangle = \langle f, xg \rangle$.

**Solution.** It’s slightly more efficient to do (b) first and then (a). To prove (b), we will use induction.

If $n = 0$, then $\phi_0 = k_0$, and if $f$ is degree 0, then $f \equiv \beta$ for some $\beta \in \mathbb{R}$, so we have

$$f = \frac{\beta}{k_0} \phi_0.$$

(I’ll also work out the $n = 1$ case directly to give more of the idea.)

Let $n = 1$, so we have $\phi_0 = k_0, \phi_1 = k_1 x + C$. Now, if $f$ is degree one then

$$f(x) = \beta_1 x + \beta_0.$$

Then if we have

$$f(x) = \alpha_0 \phi_0 + \alpha_1 \phi_1,$$

then

$$f(x) = \alpha_0 k_0 + \alpha_1 (k_1 x + C) = \alpha_1 k_1 x + (\alpha_1 C + \alpha_0 k_0),$$

so we choose $\alpha_0, \alpha_1$ to solve the two-by-two system

$$\begin{align*}
\alpha_1 k_1 &= \beta_1, \\
\alpha_1 C + \alpha_0 k_0 &= \beta_0.
\end{align*}$$

Now, we use induction. Let us assume that for any polynomial $f$ of degree $n$, we can write $f$ as a linear combination of $\{\phi_0, \ldots, \phi_n\}$. Now assume $f$ is degree $n + 1$, where

$$f(x) = \beta_{n+1} x^{n+1} + \ldots$$

Then

$$g(x) := f(x) - \frac{\beta_{n+1}}{k_{n+1}} \phi_{n+1}$$

is a polynomial of degree $n$ (since we picked constants to kill off the first term). By the induction hypothesis, we can then write

$$g(x) = \sum_{i=0}^{n} \alpha_i \phi_i,$$

so then

$$f(x) = \frac{\beta_{n+1}}{k_{n+1}} \phi_{n+1} + \sum_{i=0}^{n} \alpha_i \phi_i,$$

which is a linear combination of $\{\phi_0, \ldots, \phi_{n+1}\}$, and we are done.
Now, to prove part (a), assume \( f \) is a polynomial of degree \( k < n \). Then

\[
f(x) = \sum_{i=0}^{k} \alpha_i \phi_i,
\]

so

\[
\langle f, \phi_n \rangle = \left\langle \sum_{i=0}^{k} \alpha_i \phi_i, \phi_n \right\rangle = \sum_{i=0}^{k} \alpha_i \langle \phi_i, \phi_n \rangle = 0.
\]

Finally, we do part (c). Basically, we use the ideas above, plus a clever trick or two. First of all, we know by assumption that

\[
\phi_{n+1} = k_{n+1} x^{n+1} + O(x^n),
\]

\[
\phi_n = k_n x^n + O(x^{n-1}).
\]

From this, we know

\[
f(x) := \phi_{n+1} - \frac{k_{n+1}}{k_n} x \phi_n
\]

is a polynomial of degree \( n \). We then know, from part (b) above, that

\[
f(x) = \sum_{i=0}^{n} \alpha_i \phi_i
\]

for some constants \( \alpha_i \). Note further that since the \( \phi_i \) are orthonormal, we know that

\[
\alpha_j = \langle f, \phi_j \rangle = \left\langle \phi_{n+1} - \frac{k_{n+1}}{k_n} x \phi_n, \phi_j \right\rangle.
\]

Since \( j \leq n \), \( \langle \phi_{n+1}, \phi_j \rangle = 0 \), so that

\[
\alpha_j = -\frac{k_{n+1}}{k_n} \langle x \phi_n, \phi_j \rangle.
\]

Now, of course, we have no idea what \( \langle x \phi_n, \phi_j \rangle \) is, in general. However, we have one trick up our sleeve: notice that for any functions \( f, g \), we have

\[
\langle xf, g \rangle = \langle f, xg \rangle,
\]

because of the way we’ve defined our inner product.

**NB. Of course, we cannot do this for every inner product, but this one has a special form.**

So we then have

\[
\langle x \phi_n, \phi_j \rangle = \langle \phi_n, x \phi_j \rangle,
\]

and if \( j < n - 1 \), then the degree of \( x \phi_j \) is less than \( n \), and by part (a) this is zero. Therefore we know

\[
f(x) = \alpha_n \phi_n + \alpha_{n-1} \phi_{n-1},
\]

so define \( B_n = \alpha_n, C_n = -\alpha_{n-1} \), and we have established the formula for some \( B_n, C_n \). It remains to compute \( B_n, C_n \). We will use the two equations

\[
\langle \phi_{n+1}, \phi_n \rangle = 0, \quad \langle \phi_{n+1}, \phi_{n-1} \rangle = 0.
\]

From the first equation in (1), we have

\[
0 = \langle \phi_{n+1}, \phi_n \rangle = \langle A_n x \phi_n + B_n \phi_n - C_n \phi_{n-1}, \phi_n \rangle
\]

\[
= A_n \langle x \phi_n, \phi_n \rangle + B_n \langle \phi_n, \phi_n \rangle - C_n \langle \phi_{n-1}, \phi_n \rangle.
\]
Using the fact that
\[ \langle \phi_n, \phi_n \rangle = 1, \quad \langle \phi_{n-1}, \phi_n \rangle = 0, \]
equation (2) becomes
\[ B_n = -A_n \langle x\phi_n, \phi_n \rangle. \]
From the second equation in (1), we have
\[ 0 = \langle \phi_{n+1}, \phi_{n-1} \rangle = \langle A_n x\phi_n + B_n \phi_n - C_n \phi_{n-1}, \phi_{n-1} \rangle \]
\[ = A_n \langle x\phi_n, \phi_{n-1} \rangle + B_n \langle \phi_n, \phi_{n-1} \rangle - C_n \langle \phi_{n-1}, \phi_{n-1} \rangle. \]  
(3)
Similarly, this becomes
\[ C_n = A_n \langle x\phi_n, \phi_{n-1} \rangle. \]
We would now like to write this in terms of only \( \phi_n \). But notice that
\[ \langle x\phi_n, \phi_{n-1} \rangle = \langle \phi_n, x\phi_{n-1} \rangle = \langle \phi_n, \phi_{n-1} k_n x^{n-1} + O(x^{n-2}) \rangle \]
\[ = \langle \phi_n, k_n x^{n-1} + O(x^{n-1}) \rangle = k_n \langle \phi_n, x^n \rangle. \]
To compute the last term there, notice that
\[ 1 = \langle \phi_n, \phi_n \rangle = \langle \phi_n, k_n x^n + O(x^{n-1}) \rangle = k_n \langle \phi_n, x^n \rangle, \]
so that
\[ \langle \phi_n, x^n \rangle = \frac{1}{k_n}, \]
and thus
\[ \langle x\phi_n, \phi_{n-1} \rangle = \frac{k_n-1}{k_n} = \frac{1}{A_{n-1}}. \]

7. (Problem 2.2.10 from Keener.) **Problem fixed!** Consider the inner product
\[ \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x) \, dx \]
\((\omega(x) > 0)\).
Show that
\[ P_n(x) := \frac{1}{\omega(x)} \frac{d^n}{dx^n} (\omega(x)(1-x^2)^n) \]
is orthogonal to every polynomial of degree less than \( n \).

*Hint:* We proved this in class when \( \omega \equiv 1 \). Adapt that argument to this case.

**Solution.** Let \( f \) be a polynomial of degree \( k < n \). Then we have
\[ \langle f, P_n \rangle = \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (\omega(x)(1-x^2)^n) \, dx. \]
Recall the two lemmas we proved in class. First of all, since \((1-x^2)\) is zero at \( \pm 1 \), then if we define \( g(x) = (1-x^2)^n \), \( g \), and its first \( n-1 \) derivatives are as well, i.e.
\[ g(\pm 1) = g'(\pm 1) = g''(\pm 1) = \cdots = g^{(n-1)}(\pm 1) = 0. \]
Now, the question is, does the function \( \omega(x)g(x) \) also have the same property, i.e. are its first \( n-1 \) derivatives zero as well? The answer is yes: recall the product rule from calculus,
\[ \frac{d^p}{dx^p}(\omega(x)g(x)) = \sum_{k=0}^{p} \binom{p}{k} \omega^{(k)}(x)g^{(p-k)}(x), \]
and if we replace $p$ with any integer less than $n$, then all the derivatives on $g$ which appear in the sum are less than $n$, and these are all zero at $±1$, so therefore we know $ωg$ and its first $n−1$ derivatives are zero at $±1$.

Now we use the other lemma we proved in class, namely that if

$$h(±1) = h'(±1) = \cdots = h^{(n−1)}(±1) = 0,$$

then

$$\int_{-1}^{1} f(x) \frac{d^n}{dx^n} h(x) \, dx = (-1)^n \int_{-1}^{1} \frac{d^n}{dx^n} f(x) h(x) \, dx.$$

But notice that $f$ is a polynomial of degree $k < n$, and so if we take $n$ derivatives on $f$ it is zero.

8. (Problem 2.2.14 from Keener.) Suppose that $f(t)$ and $g(t)$ are $2\pi$-periodic functions with Fourier series representations

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt}, \quad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt}.$$  

Now define

$$h(t) = \int_{0}^{2\pi} f(t-x) g(x) \, dx.$$  

Compute the Fourier series for $h$.

**Solution.** We compute

$$h(t) = \int_{0}^{2\pi} \sum_{k=-\infty}^{\infty} f_k e^{ikt} \sum_{l=-\infty}^{\infty} g_l e^{ilx} \, dx$$

$$= \int_{0}^{2\pi} \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} e^{ilx} \, dx$$

$$= \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} \int_{0}^{2\pi} e^{ilx} \, dx.$$

Now, if $α$ is a non-zero integer, then

$$\int_{0}^{2\pi} e^{iαx} \, dx = e^{iαx} \left|_{x=0}^{x=2\pi} \right. = 0,$$

but if $α = 0$ then the integral is $2\pi$, so

$$\int_{0}^{2\pi} e^{iαx} \, dx = 2\pi δ_{α,0}.$$

Thus we have

$$h(t) = \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} \int_{0}^{2\pi} e^{i(l-k)x} \, dx$$

$$= \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} 2\pi δ_{l,k}$$

$$= \sum_{k=-\infty}^{\infty} 2\pi f_k g_k e^{ikt}.$$

So, the $k$th Fourier coefficient of $h$ is $2\pi f_k g_k$, i.e. forming the convolution of $f$ and $g$ is equivalent (up to a constant) to multiplying their Fourier series term-by-term.