Math 553. Introductory lecture
- Go over course policies, page.

Q: What is a PDE?
Why do we care?

PDE = Partial Differential Equation
-an equation that relates the partial derivatives of an unknown function.

Ex. 1
\[ u = u(x,t) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]
Heat Equation
Diffusion Equation

\[ k = \text{“diffusion coefficient”} \]

Ex. 2
\[ u = u(x,t) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]
Wave Equation

\[ c = \text{“wave speed”} \]

Think about dimensions/units:
\[ \left[ \frac{\partial u}{\partial t} \right] = \left[ \frac{u}{T} \right], \left[ \frac{\partial^2 u}{\partial x^2} \right] = \left[ \frac{u}{L^2} \right] \Rightarrow \left[ k \right] = \frac{L^2}{T} \]
\[ \left[ \frac{\partial^2 u}{\partial t^2} \right] = \left[ \frac{u}{T^2} \right], - , \left[ c \right] = \frac{L}{T}. \]

Ex. 3
\[ u_{tt} - c^2 u_{xx} + m^2 u + \gamma u P = 0. \]

1-D semilinear Klein - Gordon equation (QFT)
Def. Let $u: \mathbb{R}^n \rightarrow \mathbb{R}$, $u = u(x_1, x_2, \ldots, x_n)$. A PDE for $u$ is any equation of the form

$$F(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots) = 0$$

or

$$F(x, u, \nabla u, \nabla^2 u, \ldots) = 0.$$

If $u$ appears linearly, we say it is linear. If the highest derivative appearing in equation is semilinear, all derivatives appear linearly, but $u$ does not. If all highest-order derivatives appear linearly, but others do not, it is quasilinear.

Ex. $u_t + f(x)u_{xx} + u^3 = 0$ semilinear

Ex. Korteweg-de Vries (KdV) is quasilinear.

$$u_t + c u u_x + u_{xxx} = 0$$

Questions:

Analysis:

- Does equation have solution? Is it unique?
- Can we write solution in closed form?
- Can we approximately write solution (e.g., series)?

Qualitatively:

- Maybe cannot solve exactly, but find $\lim_{t \to \infty} u(x, t)$.
- or perhaps can slow $u(x, t) \leq f(t)$ $\forall x$. ...
We also need to discuss boundary conditions (BC) and initial conditions (IC).

**Ex. 6.** Heat equation \( u_t = k u_{xx} \), \( \Omega = (-10, 10), \ x \in \Omega, t > 0 \)

We say \( u_t = k u_{xx} \), \( x \in \Omega, t > 0 \)

\( u(-10, t) = u(10, t) = 0 \) \hspace{1cm} \text{B.C.}

\( u(x, 0) = \phi(x) \) \hspace{1cm} \text{I.C.}

Fact: This has a unique solution for "reasonable" \( \phi \),

(e.g. \( \phi \) is integrable)

I fact, \( \lim_{t \to 0} u(x, t) = 0 \) for all \( x \).

**Ex. 7.** \( u_t = k u_{xx} \), \( x \in \Omega, t > 0 \)

\( \frac{\partial u}{\partial x} (-10, t) = \frac{\partial u}{\partial x} (10, t) \)

\( u(x, 0) = \phi(x) \)

Not true that \( u \to 0 \) as \( t \to \infty \).

But: \( \lim_{t \to 0} u(x, t) = C \), where

\[
C = \frac{1}{101} \int_{-10}^{10} \phi(x) \, dx.
\]
Definitions + Notation

\[ \mathbb{Z} \] integer, \[ N = \{0, 1, 2, \ldots\} \]
\[ x \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n) \], but also
\[ (x, y) \in \mathbb{R}^n, (x, y, z) \in \mathbb{R}^3 \].

Topology.
\[ \mathbb{R}^n \] has usual metric + norm
\[ B_r(x) = \{ y \in \mathbb{R}^n \mid |x-y| < r \} \].

Def. \[ \Omega \] is a domain if it is an open connected subset of \( \mathbb{R}^n \), i.e.
\[ \forall x \in \Omega, \exists r > 0 \text{ s.t. } B_r(x) \subseteq \Omega, \]
\[ \forall y \in \Omega, \exists \Omega \subseteq [0, 1] \to \Omega \text{ continuous with } \Omega(0) = x, \Omega(1) = y \].

Def. closure of \( \Omega \) (set of all limit points) denoted \( \overline{\Omega} \).
\[ \overline{\Omega} = \mathbb{R}^n \setminus \Omega, \quad \Omega = \Omega \setminus \overline{\Omega} \].

Def. \[ f : \mathbb{R}^n \to \mathbb{R} \]. The support of \( f \) = \( \{ x \in \mathbb{R}^n \mid f(x) > 0 \} \),
closure of all points where \( f > 0 \).

Function Spaces.
\[ C(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \text{continuous} \} \]
\[ C^k(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \text{f, all derivatives of order } k \text{ or less are continuous} \} \]
\[ C(\overline{\Omega}) = \{ f \in C(\Omega) \mid \text{f can be continuously extended to } \overline{\Omega} \} \]
\[ C^k(\overline{\Omega}) \text{ similar.} \]
\[ C_b(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \text{bounded} \} \]
\[ C_0(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid \text{supp } f \text{ is compact set} \} \]
i.e. \( \{ x \mid f(x) > 0 \} \) is bounded
i.e. \( \exists R > 0 \text{ s.t. } \{ x \mid f(x) > 0 \} \subseteq B_1(0) \).
Def. If \( \Omega \) is domain, we say \( \Omega' \) is compactly supported in \( \Omega \) if \( \Omega' \subseteq \Omega \) and \( \Omega' \) is compact.

\( \Omega' \subseteq \Omega \).

Def. \( C_0(\Omega) = \{ f: \Omega \to \mathbb{R} | \exists \Omega' \subseteq \Omega, \text{supp} \ f = \Omega' \} \).
\( C^k_0(\Omega) \) similar.

Def. \( L^1(\Omega) = \{ f: \Omega \to \mathbb{R} | f \text{ is integrable} \}
\text{i.e. } \int_\Omega |f(x)| \, dx < \infty.

\( (\text{Riemann integral is fine for } 553.) \)

\( L^1_{loc}(\Omega) = \{ f: \Omega \to \mathbb{R} | \exists \Omega' \subseteq \Omega, f \in L^1(\Omega') \} \)

N.B. \( L^1(\Omega) \nsubseteq L^1_{loc}(\Omega) \)

Prop. \( C_0(\Omega) \subseteq L^1(\Omega) \), \( C(\Omega) \subseteq L^1_{loc}(\Omega) \).

Pf. \( \text{MATH 444} \)

Ex. \( \Omega = (-1,1), f(x) = \frac{1}{x-1} \)

\( f \in C(\Omega), \text{ but } f \notin L^1(\Omega) \).

But note \( -1 < a < b < 1 \), then
\( \int_a^b |f(x)| \, dx < \infty. \)
Big O, Little o, Ω notation.

Def. \( f(t) = \Theta(g(t)) \) if \( \frac{|f(t)|}{g(t)} \) is bounded in some \( t \).

\( f(t) = o(g(t)) \) if \( \frac{|f(t)|}{g(t)} \to 0 \)

\( f(t) = \Omega(g(t)) \) if \( f = \Theta(g), \ f \neq o(g) \).

E.g. if \( \lim \frac{|f(t)|}{g(t)} = C, \ 0 < C < \infty \),

then \( f = \Omega(g) \).

Ex.9. \( \sqrt{1+t^4} = \Omega(t^2) \) as \( t \to \infty \)

\( = \Omega(1) \) as \( t \to 0 \).

\( \lim_{t \to \infty} \frac{\sqrt{1+t^4}}{t^2} = \lim_{t \to \infty} \sqrt{\frac{1+t^4}{t^4}} = \lim_{t \to \infty} \sqrt{\frac{1}{t^2} + 1} = 1 \).

Review: Inverse & Implicit Function Theorems.

(see book)