Cauchy Problem / Analytic Solutions

Heat Eqn:

\[ u_t = ku_{xx} \]
\[ u(x,0) = f(x) \]

Q: How much information can we obtain from this?

\[ u_x(x,0) = f'(x) \]
\[ u_{xx}(x,0) = f''(x) \]
\[ u_{xxx}(x,0) = f'''(x) \]
\[ \vdots \]
\[ u_n(x,0) = f^{(n)}(x) \]
\[ u_{x^n}(x,0) = k u_{x^{n+1}}(x,0) = k f^{(n+1)}(x,0) \]

N.B.

\[ u_{x^n} = (u_t)^{n} = (ku_{xx})^n = k^{n} u_{x^{n+2}} \]

\[ u_{tt} = (ku_{xx})_t = ku_{xxt} = k^2 u_{xxx} \]
\[ u_{tt} x^n(x,0) = k^2 f^{(n+2)}(x) \]

\[ u_{t^n x^0}(x,0) = k a f^{(2n+2)}(x) \]

In particular, \[ u_{t^n x^0}(0,0) = k a f^{(2n+2)}(0) \]

Assume \( u(x,t) \) is analytic on \( \mathbb{R}^2 \)

(i.e. Taylor series converges everywhere uniformly)

Then: \[ u(x,t) = \sum_{a,b=0}^{\infty} \frac{u_t a x^a (0,0) t^a x^b}{a! b!} \]
\[
\sum_{a,b=0}^{\infty} \frac{f(a+b)(0)}{a!b!} k^a t^b x^b
\]

**Ex. 1.**
\[f(x) = x^4. \quad f'(x) = 4x^3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x, \quad f''''(x) = 24, \quad 0 - \]
\[\text{at } x = 0: \quad f = f' = 0, \quad f'' = 2, \quad f''' = 0 - \]
\[\Rightarrow \text{only deriv ii is 2.} \quad 2a + b = 2.\]
\[a = 1, b = 0 \quad \frac{2}{1!0!} k t = 2kt\]
\[a = 0, b = 2 \quad \frac{2}{2!0!} x^2 = x^2\]

Check:
\[U_t = 2kt, \quad U_{xx} = 2kUxx \quad \checkmark\]

**Ex. 1.**
\[f(x) = x^3 \quad f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6, \quad 0 - \]
\[\Rightarrow \quad 2a + b = 3.\]
\[a = 1, b = 1 \quad \frac{6}{1!1!} k t = x^3 + 6kt.\]
\[a = 0, b = 3 \quad \frac{6}{3!} x^3.\]

**Ex. 3.**
\[f(x) = x^4. \quad f^{(n)}(0) = \begin{cases} 6! & n = 0 \Rightarrow 2a + b = 6. \\ 0 & \text{else.} \end{cases}\]
\[a = 3, b = 0 \quad \frac{6!}{3!0!} k^3 t^3 = 120 k^3 t^3\]
\[a = 2, b = 2 \quad \frac{6!}{2!2!} k^2 t^2 x^2 = 180 k^2 t^2 x^2\]
\[a = 1, b = 4 \quad \frac{6!}{1!4!} k t x^4 = 30 k t x^4\]
\[a = 0, b = 4 \quad \frac{6!}{6!} x^4 = x^4.\]
\[ U(x,t) = x^6 + 30kt x^4 + 180 k^2 t^2 x^2 + 120 k^3 t^2. \]

Check: \[ U_t = 30k x^4 + 360 k^2 t x^2 + 360 k^3 t^2. \]

\[ U_{xx} = 30 x^4 + 360 k + x^2 + 360 k^2 t^2 \]

Q: How do we know this will work?

\[ U_t = k u_{xx}, \quad U(0,t) = g(t). \]
\[ U_t = g'(t) \quad \text{(??)} \]
\[ U_{xx}(0,t) = \frac{1}{k} U_t = \frac{1}{k} g'(t) \]

This problem is under-specified, ??!

\[ U_{xx} + u_{yy} - u_y = u^3 \]
\[ U(x,0) = x \]
\[ U_y(x,0) = x^2 \]

First, \[ U_x(x,0) = 1, \quad U_{xx} = 0, \quad 0, \quad 0 \]
\[ U_{yx}(x,0) = 2x, \quad U_{xxy}(x,0) = 2, \quad U_{xxyx} = 0, \quad \]
\[ U_{yy} = ? \]

\[ U_{yy} = u^3 + u_y - u u_{xx} \]
\[ U_{yy}(x,0) = x^3 + x^2 - 0 = x^2 + x^2 \]
\[ U_{yyxx}(x,0) = 6x + 2 \]
\[ U_{yyxy}(x,0) = 6 \]
\[ U_{yyxyy}(x,0) = 0 \]
\[ U_{yy} = \frac{\partial}{\partial y} (U_{yy}) = \frac{\partial}{\partial y} (u^2 + uy - u u_{xx}) \]

\[ = 3u^2uy + uy^2 - uy u_{xx} - u u_{yxx}. \]

**Note:** Every term on RHS has 2 or fewer y's.

\[ = 3x^2x^2 + x^3 + x^3 - 0 - 2x \]
\[ = 3x^4 + x^3 + x^2 - 2x. \]

\[ U_{yyyy} = |2x^2 + 3x + 2x - 2|, \quad \text{et al.} \]

\[ U_{yy} = \frac{\partial}{\partial y} (1) = \ldots \]

Every term will have 3 or fewer y's, et al.

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**General Theory.**

**Def.** A multi-index is \( \alpha \in \mathbb{N}^n \), \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

\[
|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_n!.
\]

\( \forall x \in \mathbb{R}^n, \)

\[ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}. \]

Then, \( m^{th} \) order equation is

\[ F(x, D^\alpha u) = 0, \quad |\alpha| \leq m. \]
Assume we can write our problem as

\[
\frac{\partial^m u}{\partial x_n^m} = G(x, \partial^\alpha u), \quad |\alpha| \leq m, \quad \alpha \neq (0, 0, \ldots, 0)
\]

AND

on \( S = \{ x \in \mathbb{R}^n \mid x_n = 0 \} \approx \mathbb{R}^{n-1} \), we have specified

\[
u \big|_{x_n = 0} = g_0, \quad \frac{\partial^l u}{\partial x_n^l} \big|_{x_n = 0} = g_l, \quad \ldots, \quad \frac{\partial^{m-1} u}{\partial x_n^{m-1}} \big|_{x_n = 0} = g_{m-1}.
\]

Then we can determine all partial derivatives on the whole of \( S \).

**How?** Let \( \alpha \in \mathbb{N}^n \).

If \( \alpha_n = 0 \), differentiate \( g_0 \) appropriately, i.e.

\[
U_{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}} = x_n^{\alpha_n-1}(x_1, \ldots, x_{n-1}, 0) = g_0 x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} (x_n - x_n^{\alpha_n}).
\]

If \( \alpha_n = 1 \), differentiate \( g_1 \).

\[
U_{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}} = x_n^{\alpha_n-1}(x_1, \ldots, x_{n-1}, 1) = g_1 x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}.
\]

If \( \alpha_n = k \), differentiate \( g_k \), \( k = 0, 1, \ldots, m-1 \).

If \( \alpha_n = m \), use PDE.

\[
U_{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}} = (U_{x_n^m})_{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n-1}} = (G(x, \partial^\alpha u))_{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n-1}}.
\]

Note: all derivatives have \((m-1)\) or fewer on \( x_n \).
$\alpha(\alpha_1, \ldots, \alpha_n, \ldots, \alpha_{n+1})$

$\alpha_{n+1} = (D_{\alpha_n} G(x, D^\alpha u)) \alpha_1 \ldots \alpha_n$

Hat in a fewer deriv in \( \kappa_\alpha \) direction,

e.t.

More generally,

Defn. Let \( S \) be \((n-1)\)-dim hypersurface in \( \mathbb{R}^n \), consider PDE of the form \( F(x, D^\alpha u) = 0, \ |\alpha| \leq m \).

If \( S \) has the property that \( u|_S, \ \frac{\partial u}{\partial v}|_S, \ \frac{\partial^2 u}{\partial v_1 \partial v_2}|_S, \ldots, \frac{\partial^m u}{\partial v_1 \ldots \partial v_m}|_S \) determine all derivatives in \( S \), we say \( S \) is non-characteristic for the PDE.

N.B. If we can find a C.O.V. \( S \rightarrow \{x_n = 0\} \subset \mathbb{R}^n \), and PDE becomes \( \sum_{\alpha} \chi^\alpha = G(x, D^\alpha u), \ |\alpha| \leq m, \ \alpha \neq (0, \ldots, 0) \)

then \( S \) is non-characteristic.

Thm. (Cauchy-Kovalevskaya). Write PDE in normal form (2). If \( g_j \) are real analytic in nbhd of \( 0 \in \mathbb{R}^{n-1} \), and \( G \) is real analytic in nbhd of \( (0, D^\alpha u(0)) \), then \( \exists! \) real analytic solution of (2) defined in an open nbhd of \( 0 \in \mathbb{R}^n \).

PF. not here.
Remark. This is great news! We can always generate solutions of PDE with enough work!

Remark. Wait, not so fast. What if our solution is not analytic? What if initial data not analytic?? (Recall shock waves)

Even worse (perhaps), existence isn’t enough for physical meaningfulness. For example, let \( h, h_1, h_2, \ldots, h_k, \ldots \) be sequence of functions s.t. \( \lim_{k \to \infty} h_k(x) = h(x) \).

Let \( u_k(x,t) \) solve \( F(x,t,u,u_x) = 0 \) with data \( h_k \).

Let \( u \) solve with data \( h \).

Is \( \lim_{k \to \infty} u_k(x,t) = u(x,t) \)?

Remark. Analyticity is necessary; \( C^0 \) is too weak.

See book.