1.2. Shock and Rarefaction Waves

Def.: A semi-linear PDE is said to be a conservative law if it can be written

\[ \frac{\partial}{\partial x} (G(u)) + \frac{\partial}{\partial y} u = 0; \]

where \( G \) is a given function.

Integrate (1) on \( x \in [a, b] \):

\[ \int_a^b \frac{\partial}{\partial x} G(u(x,y)) \, dx + \int_a^b \frac{\partial}{\partial y} u(x,y) \, dx = 0 \]

(2) \[ G(u(b,y)) - G(u(a,y)) + \frac{\partial}{\partial y} \int_a^b u(x,y) \, dx = 0 \]

This (2) is called the weak form of (1).

(Weak formulation, weak version, etc.)

Why? If \( u \in C^1(\mathbb{R}^2, \mathbb{R}) \), \( G \) is smooth and \( u \) satisfies (2), then \( u \) also satisfies (1) by differentiating. However, note that (2) makes sense for a broader class of functions than (1).

Def. If \( u(x,y) \) satisfies (2), but is not \( C^1 \), we say \( u \) is a weak solution of (1).

Ex.1: \( u(x,y) = \begin{cases} 0 & x < 0 \\ \gamma y & x \geq 0 \end{cases} \)

\[ \frac{\partial}{\partial y} \int_a^b u(x,y) \, dx \text{ exist. a.e. } b. \]
Let us assume that we have a solution with one point of discontinuity for each \( y \) (but the location can depend on \( y \)).

Ex. \( y = 0 \) \hfill \( y = 1 \)

\[ u(x, y) \]

Let \( \xi(y) \) be the location of the discontinuity. Take \( a < \xi(y) < b \); plug in for (2).

Define

\[ U_R(y) = \lim_{x \to \xi(y)^+} u(x, y), \quad U_L(y) = \lim_{x \to \xi(y)^-} u(x, y). \]

(2) becomes

\[ G(u(b, y)) - G(u(a, y)) + \frac{d}{dy} \int_{a}^{b} u(x, y) \, dx + \xi'(y)(u(\xi(y)) - u(y)) \]

\[ + \int_{\xi(y)}^{b} u(x, y) \, dx \]

Leibnitz Rule:

\[ \frac{d}{dy} \int_{a}^{b} u(x, y) \, dx = \int_{a}^{b} u_y(x, y) \, dx + \xi'(y)u(\xi(y)) - u(y) \]

(\text{HW}) where \( u(z - y) = \lim_{x \to z^-} u(x, y) \).

Similarly,

\[ \frac{d}{dy} \int_{\xi(y)}^{b} u(x, y) \, dx = \int_{\xi(y)}^{b} u_y(x, y) \, dx + \xi'(y)u(\xi(y)) - u(y). \]

Then (2) \Rightarrow

\[ G(u(b, y)) - G(u(a, y)) + \xi'(y)(u(\xi(y)) - u(y)) \]

\[ + (\int_{a}^{b} + \int_{\xi(y)}^{b}) u_y(x, y) \, dx \]

Let \( b \to \xi(y)^+, \quad a \to \xi(y)^- \).
\[ G(u_R(y)) - G(u_L(y)) + \frac{G'(y)}{2} (u_L(y) - u_R(y)) = 0 \]

\[ \Rightarrow \left\{ \begin{align*} 
G'(y) &= \frac{G(u_R) - G(u_L)}{u_R - u_L} \quad \text{(Rankine-Hugoniot)} 
\end{align*} \right. \]

**Note:** Knowing only local information, we obtain the speed of the shock (discontinuity).

Read Burgers' equation \( u u_x + u y = 0 \). \( G(u) = \frac{1}{2} u^2 \).

To:
\[ u(x,0) = \begin{cases} 
  u_0 & x < 0 \\
  0 & x > 0 
\end{cases} \]

At \( y = 0 \), discontinuity at 0, so \( \xi(0) = 0 \).

How does \( \xi(y) \) move?
\[ u_R(0) = 0 \quad G(u_R(0)) - G(u_L(0)) = 0 - \frac{1}{2} u_0^2 = \frac{u_0}{2} \]
\[ u_L(0) = \phi(0) \quad u_R(0) - u_L(0) \]
\[ \xi'(0) = u_0 \]

But note: we always have \( u_R = 0, u_L = 0 \), so
\[ \xi'(y) = u_0 \quad \xi(0) = 0 \Rightarrow \xi(y) = \frac{u_0}{2} y \]
\[ u(x,y) = \begin{cases} 
  u_0 & x < u_0 y/2 \\
  0 & x > u_0 y/2 
\end{cases} \]

Check: \( u(x,0) \). Choose any \( y \), then choose \( x < \frac{u_0}{2} y \).

\[ G(u(0,y)) = G(0) = 0 \]
\[ G(u(a,y)) = G(u_0) = \frac{1}{2} u_0^2. \]

\[ \int_a^b u(x,y) \, dx = \int_a^{u_0 y / 2} u_0 \, dx + \int_{u_0 y/2}^b 0 \, dx = \left( \frac{u_0 y}{2} - a \right) u_0. \]

\[ \frac{d}{dy} \int_a^b u(x,y) \, dx = \frac{u_0 y}{2}. \]

\[ 0 - \frac{u_0^2}{2} + \frac{u_0^2}{2} = 0 \quad \checkmark \]

Same equation, different I.C.

\[ u(x,0) = \begin{cases} 
  u_0 & x \leq 0 \\
  u_0(1-x) & x \in (0,1) \\
  0 & x \geq 1 
\end{cases} \quad \text{I.C. is continuous.} \]

So no shock at \( t = 0 \).
Draw characteristics.

Recall: formula for characteristic is \( (h(x)y + x, y) \).

If \( x > 1 \), \((x, y) \uparrow\)

\( x \leq 0 \), since \((u_0y + x, y)\)

This hits vertical char. when \( u_0y + x = 1 \).

Consider \( x = 0 \). \( u_0y = 1 \) \( \Rightarrow y = 1/u_0 \)

Chat if \( x \in (0, 1) \)? \( h(x) = u_0(1-x) \)

\( u_0(1-x)y + x = 1 \)
\( u_0(1-x)y = 1 - x \)
\( y = 1/u_0 \) all crash at same place.
Shock forms at $x = 1$, $y = \frac{1}{2}u_0$.

Again, $u_L = u_0$, $u_R = 0$, so $\xi(y) = \frac{u_0}{2}$.

$s$ $(1) = \frac{1}{u_0}$, so

$s(y) = \frac{1}{u_0} + \frac{u_0}{2} y$.

**Snapshots.**

$y = 0$

$y = \frac{1}{2} u_0$

$y = 0.99 \frac{1}{u_0}$

$y = \frac{1}{u_0}$

then shock moves at velocity $\frac{u_0}{2}$. 
Rarefaction Wave.

IC. \[ u(x, 0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \]

characteristics.

Two (of many) possibilities.
1) add shock in the middle. \( \Phi(y) = \frac{y}{2} \)
2) linear interpolation.

1) \( u(x, y) = \begin{cases} 0 & x < y/2 \\ 1 & x > y/2 \end{cases} \) shock

2) \( u(x, y) = \begin{cases} 0 & x < 0 \\ \frac{x}{y} & 0 < x < y \\ 1 & x > y \end{cases} \) rarefaction

Claim: Both \( u_1, u_2 \) satisfy (2).

Checks: If \( a < y/2 \) or \( y/2 < b < c \), all terms \( n \to \) cancel.

If \( a < y/2 < b \),

\[ G(u(b, y)) = G(1) - \frac{1}{2} \]

\[ G(u(a, y)) = G(-1) = 0 \]

\[ \frac{d}{dy} \int_a^b u(x, y) \, dx = \frac{d}{dy} \left( \int_a^{y/2} 1 \, dx \right) - \frac{d}{dy} \left( b - \frac{y}{2} \right) = -\frac{1}{2} \]
If \( a < b < 0 , \ y < a < b , \) all term cancel.

say \( a < 0 , \ b > y. \)

\[
G(u(b,y)) = G(1) = \frac{1}{2}
\]

\[
G(u(a,y)) = G(0) = 0
\]

\[
\frac{d}{dy} \int_a^b u(xy) \, dx = \frac{d}{dy} \left( \int_0^y \frac{x}{y} \, dx + \int_y^b 1 \, dx \right) = -\frac{1}{2} \quad \checkmark
\]

\( a < 0 , \ 0 < b < y. \)

\[
G(u(b,y)) = G(b/y) = \frac{b^2}{2y^2}
\]

\[
\frac{d}{dy} \int_a^b u(xy) \, dx = \frac{d}{dy} \int_0^y \frac{x}{y} \, dx = \frac{d}{dy} \frac{b^2}{2y} = \frac{-b^2}{2y^2}.
\]

Both work!! Which is "correct"?

**Physical argument:** never form shocks unless you have to!

(\text{minimize entropy} \ldots)

\[= \text{rarefaction wave}.
\]

\underline{Traffic Flow}.

Let \( p(x,t) \) = density of cars at \( x,t \)

\[ q(x,t) = \text{flow rate at } x,t
\]
\[ \int_{x_1}^{x_2} q(x,t) \, dx = \frac{d}{dt} \int_{x_1}^{x_2} q(x,t) \, dx \quad \text{in} \quad (x_1, x_2) \]

\[ \frac{d}{dt} \int_{x_1}^{x_2} p(x,t) \, dx = q(x_1,t) - q(x_2,t) = -\frac{d}{dx} \int_{x_1}^{x_2} q(x,t) \, dx \]

\[ \int_{x_1}^{x_2} (p_t + q_x) \, dx = 0. \quad \text{Since hold!} \quad x_1, x_2, \]

\[ p_t + q_x = 0. \quad \text{(flux equation)} \]

If \( q = G(p) \), then \( p_t + (G(p))_x = 0 \) is conservation law.

E.g., \( G(p) = \)

\[ \frac{\text{max}}{\text{bump}} \]

"bump" to "bump"