Section 1.1.

Cauchy Problem for Quasi-linear PDE: \( x \).

**Ex 1.**

\[ u_t + a u_x = 0 \quad x, t \in \mathbb{R} \]

\[ u(x, 0) = h(x) \]

Let \( x(t) \) be curve, can we choose it so that \( u(x(t), t) \) is constant along this curve?

\[
\frac{d}{dt}(u(x(t), t)) = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial t}.
\]

If \( \frac{dx}{dt} = a \), then

\[
\frac{d}{dt}(u(x(t), t)) = 0.
\]

\[ \Rightarrow \quad x(t) = at + x(0). \quad \text{(2) } x - at = \text{const}. \]

1. Along the curve \( x - at = \text{const} \), the function is constant.

\[ \Rightarrow \quad u(x(t), t) = h(x - at). \]

Check:

\[
\frac{\partial u}{\partial x} = h'(x - at),
\]

\[
\frac{\partial u}{\partial t} = -a \cdot h'(x - at).
\]

1. General: \( a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} = c(x, y, u). \)

Let \( u(x, y) \) be solution, consider the graph

\[ z = u(x, y). \]
\[ N = (-u_x, -u_y, 1) \]

Note that: \[(a, b, c) \cdot (-u_x, -u_y, 1) = -au_x - bu_y + c = 0.\]

\[ V = (a, b, c) \] is tangent to the surface.

A vector field in \( \mathbb{R}^3 \). (Since \( a = a(x, y, z), \) etc.)

**Definition:** A surface in \( \mathbb{R}^3 \) that is tangent everywhere to a given vector field \( V: \mathbb{R}^3 \to \mathbb{R}^3 \) is called an integral surface of \( V \).

**Problem:** Given a curve \( \Gamma \) in \( \mathbb{R}^3 \), can we find a solution to PDE whose graph contains \( \Gamma \)?

**Vector Field:** \( V = (a, b, c) \). Integrate there!

\[ \frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z). \]

**Characteristic Equations.**

**Theorem (441):** If \( a, b, c \) are all \( C^1 \), then (2) has a unique solution for some \( (t_0, t_0) \) small enough, given \( x(t_0) = x_0, \ y(t_0) = y_0, \ z(t_0) = z_0 \).

By varying the I.C., we can fill out a surface of solutions under one condition.
If \( \Gamma \) is a noncharacteristic curve (i.e. \( \Gamma \) is nowhere tangent to \( V \)), then the v.f. \( V \) admits a unique integral surface \( S' \) containing \( V \).

**Pf.**

Write \( \Gamma = (f(s), g(s), h(s)) \) parameterized by \( s \in \mathbb{R} \).

At, solve (2) with I.C. \( x_0 = f(s), y_0 = g(s), z_0 = h(s) \).

Consider the maximal \( t < 0 < t' \) such that the ODE has a solution, giving a curve. The union of these curves defines a surface, call it \( S' \). By E-U Theorem of ODE, \( S' \) is unique.

**Potential challenges.**

1. Hard to solve (2).
2. Even if we have \( S' \) parameterized in terms of \( s, t \), can we write \( t \) in terms of \( x, y \)? (i.e. can we invert the function \( x = x(s, t), etc. \))

**Semilinear.**

(3) \[ a(x, y) \, dx + b(x, y) \, dy = c(x, y, u). \]

(4) \[
\begin{align*}
\frac{dx}{dt} &= a(x, y) \\
\frac{dy}{dt} &= b(x, y) \\
\frac{dz}{dt} &= c(x, y, z) \\
\end{align*}
\]

\( x(s, 0) = f(s), \ y(s, 0) = g(s), \ z(s, 0) = h(s). \)

\( x(s, t), \ y(s, t), \ z(s, t) \)
N.B. First two equations decouple (no $z$'s appear), so solve them first. Then plug in.

Then if we can write $s = s(x, y), \ t = t(x, y)$, then

$$z(s, t) = z(s(x, y), t(x, y)) = u(x, y)$$

Q: What do we need to obtain $s(x, y), t(x, y)$?
We can do this if

$$\det \begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} \neq 0. \quad \text{(Inverse exists)}$$

at $t = 0$, we have

$$\det \begin{pmatrix} f'(s) & a(f(s), g(s)) \\ g'(s) & b(f(s), g(s)) \end{pmatrix} \neq 0.$$

If this determinant is $0$ at $t = 0$, it must be $0$ for some open set of $t$'s containing $t = 0$. Therefore there funtions are nowhere in some neighborhood of $\Gamma$.

Ex 2. $u_x + 2u_y = u^2$.

$u(x, 0) = h(x)$.

$\Gamma = (s, 0, h(s))$.

$f(s) = s, \ g(s) = 0 \quad a = 1, \ b = 2$.

$$\det \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = 2 \neq 0.$$
Q: Is L characteristic?
\[(a, b, c) = (1, 2, 1). \square\]

Target to \( L \) is \((1, 0, h'(s))\).
There are never collinear!

NB (All we need here is the coefficient of \( y' \neq 0 \))

\[
\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2 \quad \frac{dz}{dt} = z^2
\]

\[
x(s, 0) = s \quad y(s, 0) = 0 \quad z(s, 0) = h(s)
\]

\[
x(s, t) = s + t \quad y(s, t) = 2t \quad z(s, t) = ?
\]

\[
\frac{dz}{dt} = z^2 \quad \frac{dz}{z^2} = dt \quad \frac{-1}{z} = t + C
\]

\[
z(t) = \frac{-1}{t+C}.
\]

Plug in \( t=0 \),

\[
z(0) = \frac{-1}{C+0} \Rightarrow C = -\frac{1}{z(0)}.
\]

\[
z(t) = \frac{-1}{t+C} = \frac{-z(0)}{t-z(0)}
\]

\[
x(s,t) = \frac{-1}{t-z(0)} = \frac{-z(0)}{t^2-1} = \frac{z(0)}{1-t^2}.
\]

\[
z(s(x,y), t(x,y)) = \frac{h(x-y/2)}{1-\frac{y}{2}h(x-y/2)}. \quad \text{NB: denominator can = 0, can blow up!}
\]

\[
u(x,y) = \frac{h(x-y/2)}{1-\frac{y}{2}h(x-y/2)}.
\]
Ex 3. Inviscid Burger Equation. (A-L)

Model: population of particles on line, each moving with constant velocity.

\( u(x, y) \) = velocity of particle at space \( x \), time \( y \).
\( \dot{x} = u(x', y) \) = position of particle.
\( u(x, y) \) must be constant (in \( y \)).

\[ \frac{dy}{dt} u(x(y), y) = \frac{dx}{dy} \frac{du}{dx} + \frac{du}{dy} = 0 \]

\( u \left( u_x + u_y = 0 \right) \) or \( \dot{y} + \frac{1}{2} (u^2)_x = 0 \)

\( u(x, 0) = h(x) \) \( \Rightarrow \) initial velocity profile.

\[ \begin{cases} \frac{dx}{dt} = 2, & \frac{dy}{dt} = 1, & \frac{dz}{dt} = 0 \\ x(s, 0) = s, & y(s, 0) = 0, & z(s, 0) = h(s) \end{cases} \]

\( z(s, t) = h(s), \ y(s, t) = t \).

\( x(s, t) = h(s) - t h'(s) + x(s, 0) \)

\( = t h(s) + s \).

\[ \begin{pmatrix} x(s, t) \\ y(s, t) \end{pmatrix} = \begin{pmatrix} 1 + t h'(s) \\ h(s) \end{pmatrix} = 1 + t h'(s). \]

At \( t = 0 \), \( \text{det} = 1 \neq 0 \) so invertible.

But \( t = -1/\lambda h'(s) \), could get regularity.

Say \( h(x) \)

This is velocity profile!

\[ \text{What happens if this comes next?} \]