1. (McOwen 5.1.1). A square two-dimensional plate of side length \( a \) is heated to a uniform temperature \( U_0 > 0 \). At \( t = 0 \), all sides are reduced to zero temperature. Describe the temperature profile \( u(x, y, t) \) for all \( t > 0 \).

Solution: This is a problem with Dirichlet boundary conditions. We know that the eigenfunctions of the Laplacian are

\[
\phi_{kl}(x, y) = \sin(k\pi x/a) \sin(l\pi y/a), \quad \lambda_{kl} = \frac{\pi^2}{a^2}(k^2 + l^2).
\]

Thus our solution to the heat equation is

\[
u(x, y, t) = \sum_{k,l} A_{k,l} e^{-\lambda_{kl}t} \phi_{kl}(x, y),
\]

and it only remains to determine \( A_{k,l} \). Plugging in \( t = 0 \) gives

\[U_0 = \sum_{k,l} A_{k,l} \phi_{kl}(x, y).\]

Since the eigenfunctions separate, it might make sense to guess that the coefficients separate, i.e. \( A_{k,l} = B_k C_l \). We then have

\[
\sum_k B_k \sin(k\pi x/a) \sum_l C_l \sin(l\pi y/a) = U_0.
\]

Using Fourier series, we see that we can choose

\[
B_k = \sqrt{\frac{4a}{k\pi}}, \quad k \text{ odd},
\]

and similarly for \( C_l \), so we have

\[
A_{k,l} = \frac{16U_0 a^2}{\pi^2 kl}, \quad k, l \text{ odd}.
\]

2. (McOwen 5.1.7). Assume that \( u \) satisfies the heat equation in \( \Omega \), and define the energy of \( u \) by

\[
E(t) = \int_{\Omega} u^2(x, t) \, dx.
\]

(a) Let \( U = \Omega \times (0, \infty) \) and assume \( u \in C^2_x C^1_t(U) \). If \( u = 0 \) on \( \partial\Omega \), show that \( E \) is nonincreasing as a function of \( t \).

(b) Let \( U = \Omega \times (0, \infty) \) and assume \( u \in C^2_x C^1_t(U) \). If \( \partial u/\partial \nu = 0 \) on \( \partial\Omega \), show that \( E \) is nonincreasing as a function of \( t \).

(c) Use the previous results to show uniqueness in \( C^2_x C^1_t(U) \) for the Dirichlet or Neumann heat equations.
Solution: Let us compute the evolution of $E$, and we see

$$\frac{d}{dt} E(t) = \partial_t \int_{\Omega} u^2(x, t) \, dx = \int_{\Omega} uu_t \, dx = \int_{\Omega} u \Delta u \, dx.$$  

Using Integration by Parts, we get

$$\frac{d}{dt} E(t) = -\int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \, dS_x.$$  

Under either Dirichlet or Neumann boundary conditions, the second term is zero. Clearly, the first term is either negative or zero. Also note that $E = 0$ implies $u = 0$ almost everywhere.

Now for uniqueness. Let us say that $u, v$ satisfy the heat equation on $\Omega$ and the same boundary conditions (either Dirichlet or Neumann) on $\partial\Omega$. Then $w = u - v$ satisfies the heat equation with zero boundary conditions, which means that $E_w(0) = 0$, which means that $E_w(t) = 0$ for all $t > 0$, which means that $w \equiv 0$ except perhaps for a set of measure zero.

3. **(McOwen 5.2.1).** Recall the one-dimensional version of this problem from September! Let $K(x, y, t)$ be the Gaussian heat kernel.

(a) Show that $K$ is $C^\infty$ and satisfies $(\partial_t - \Delta_x)K(x, y, t) = 0$ for $x, y \in \mathbb{R}^n, t > 0$.

(b) Prove that

$$\int_{\mathbb{R}^n} K(x, y, t) \, dx = 1$$

for $x, y \in \mathbb{R}^n, t > 0$.

(c) For any $\delta > 0$, show that

$$\lim_{t \to 0^+} \int_{|x-y|>\delta} K(x, y, t) \, dx = 0,$$

and that this limit is uniform for $x \in \mathbb{R}^n$.

(d) Assuming $g \in C_B(\mathbb{R}^n)$, show that $u(x, t)$ defined by

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t)g(y) \, dy$$

has the property that

$$\lim_{t \to 0^+} u(x, t) = g(x).$$

Solution:

(a) This can be obtain by direct differentiation.
(b) We have
\[
\int_{\mathbb{R}^n} e^{-|x-y|^2/4t} \, dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n}(x_i-y_i)^2/4t} \, dx = \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-(x_i-y_i)^2/4t} \, dx.
\]

Writing \(dx = dx_1 \ldots dx_n\), these integrals separate and we obtain
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-(x_i-y_i)^2/4t} \, dx = \left( \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \, dx \right)^n = \sqrt{4\pi t}.
\]

(c) We have
\[
\begin{aligned}
4. \quad & \text{(McOwen 5.2.2). Let } u(x,t) \text{ be defined by } [1], \text{ and let } g \in C_B(\mathbb{R}^n). \\
& \text{(a) Show that } |u(x,t)| \leq \sup_{y \in \mathbb{R}^n} |g(y)|. \\
& \text{(b) Show that if } g \in L^1(\mathbb{R}^n), \text{ then } \\
& \quad \lim_{t \to \infty} u(x,t) = 0 \\
& \quad \text{uniformly for } x \in \mathbb{R}^n.
\end{aligned}
\]

Solution:

(a) We have
\[
|u(x,t)| \leq \int_{\mathbb{R}^n} |K(x,y,t)g(y)| \, dy \leq \sup_{y \in \mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |K(x,y,t)| \, dy
\]

Noting that \(K > 0\), we can remove the absolute value and the last integral is exactly 1.

(b) 5. \quad \text{(McOwen 5.2.4). Formally check that}
\[
u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} d^{k+1}t^{k+1} e^{-1/t^2}
\]
satisfies \(u_t = u_{xx}\) for \(x \in \mathbb{R}, t > 0\) and that \(u(x,0) = 0\).

Solution: The first part is easy, the second part is harder.

We have
\[
u_t = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} d^{k+1}t^{k+1} e^{-1/t^2}
\]
and

\[ u_{xx} = \sum_{k=0}^{\infty} \frac{1}{(2(k-1))!} x^{2(k-1)} \frac{d^k}{dt^k} e^{-1/t^2} \]

and it is clear that these are the same with a different summation variable.

Now, to show that \( u(x,0) = 0 \). This is clearly equivalent to showing that

\[ \left. \frac{d^k}{dt^k} e^{-1/t^2} \right|_{t=0} = 0. \]

Let us define

\[ f_k(t) = \frac{d^k}{dt^k} e^{-1/t^2}, \quad g_k(t) = e^{1/t^2} f_k(t). \]

Then we have

\[ f_{k+1}(t) = (f_k(t))' = (e^{-1/t^2} g_k(t))' = e^{-1/t^2} g'_k(t) + e^{-1/t^2} (-2t^{-3}) g_k(t) = e^{-1/t^2} (g'_k(t) - 2t^{-3} g_k(t)), \]

and thus we have the recursion relation

\[ g_{k+1} = g'_k(t) - 2t^{-3} g_k(t), \quad g_0(t) = 1. \]

We can compute a few of these directly:

\[ g_1(t) = -2t^{-3}, \quad g_2(t) = 4t^{-6} + 6t^{-4}, \ldots \]

It is not hard to see that \( g_k(t) \) will be a polynomial in \( t^{-1} \) with highest power \( 3k \). Let us write \( p_k(s) = g_k(1/s) \). From this we can deduce that

\[ \lim_{t \to 0^+} f_k(t) = \lim_{t \to 0^+} e^{-1/t^2} g_k(t) = \lim_{s \to \infty} \frac{p_k(s)}{e^{s^2}} = 0. \]