1. Show that the fundamental solution $\psi(r)$,

$$
\psi(r) = \begin{cases} 
  c_1 + c_2 \log r, & n = 2, \\
  c_1 + c_2 r^{2-n}, & n \geq 3,
\end{cases}
$$

is integrable near the origin. Deduce from this that $\psi \in L^1_{loc}(\mathbb{R}^n)$.

**Solution:**
Let $\delta > 0$, and we want to show that the integral of $u(x) = \psi(|x|)$ over $B_\delta(0)$ is finite. We can choose $c_1 = 0$ wlog since the integral of a constant on a bounded domain is finite.

We can write

$$
\int_{B_\delta(0)} u(x) \, dx = \int_0^\delta dr \int_{\partial B_r(0)} dS_\theta u(\theta).
$$

Noting that $u$ is constant on $\partial B_r(0)$, we obtain

$$
\int_{B_\delta(0)} u(x) \, dx = \int_0^\delta drr^{n-1}\psi(r)\omega_n.
$$

If $n = 2$, the integrand is $r \log r$ and if $n > 2$ it is $r$. Clearly these are integrable (and in fact are not even singular at the origin).

2. (McOwen 4.2.1).

**Solution:** We start with Poisson’s integral formula

$$
u(\xi) = a^2 - |\xi|^2 \int_{|x|=a} \frac{g(x)}{|x-\xi|^2} \, dS_x.
$$

When $n = 2$, $a = 1$, this integral is over the circle of radius 1, and we have $\omega_2 = 2\pi$. Writing $\xi = re^{i\theta}$ and $x \in S^1 = e^{i\psi}$, we have

$$
|x-\xi|^2 = |(r \cos \theta - \cos \psi, r \sin \theta - \sin \psi)|^2
= r^2 \cos^2 \theta - 2r \cos \theta \cos \psi + \cos^2 \psi + r^2 \sin^2 \theta - 2r \sin \theta \sin \psi + \sin^2 \psi
= r^2 + 1 - 2r(\cos \theta \cos \psi - \sin \theta \sin \psi) = r^2 + 1 - 2r \cos(\theta - \psi),
$$

and the result in part (a) follows.

3. (McOwen 4.2.10).

**Solution:**
(a) We again start with Poisson’s integral formula

\[ u(\xi) = \frac{a^2 - |\xi|^2}{a\omega_n} \int_{|x|=a} \frac{g(x)}{|x-\xi|^n} dS_x, \]

where \( g \) is the boundary condition on the sphere. Since we have a harmonic \( u \) and a ball \( B_0(0) \) strictly inside the domain, then the function \( g \) is given by the function \( u \) itself on the boundary.

Recall the two forms of the triangle inequality

\[ |x - y| \leq |x| + |y| = |x| + |y|, \quad |x - y| \geq |x| - |y|. \]

Using the second of these, we have

\[ |x - \xi|^n \geq (|x| - |\xi|)^n = (a - |\xi|)^n, \]

and thus

\[ \int_{|x|=a} \frac{u(x)}{|x-\xi|^n} dS_x \leq \int_{|x|=a} \frac{u(x)}{(a - |\xi|)^n} dS_x = \frac{1}{(a - |\xi|)^n} \int_{|r|=a} u(x) dS_x. \]

Now recall the Gauss Mean Value Theorem, which says that

\[ \int_{|r|=a} u(x) dS_x = a^{n-1} \omega_n u(0), \]

i.e. that the value at the center of the sphere is equal to (a suitably rescaling) average on the sphere. Putting this together gives

\[ u(\xi) \leq \frac{a^2 - |\xi|^2}{a\omega_n} \frac{1}{(a - |\xi|)^n} a^{n-1} \omega_n u(0) = \frac{a^{n-2} (a + |\xi|)}{(a - |\xi|)^n} u(0), \]

giving the right-hand inequality. Similarly, using the other form of the triangle inequality gives the left-hand inequality.

(b) We now use this to prove the Harnack Inequality. As a lemma, we will first show that if \( B_r(x_0) \) is any ball such that \( u \) is harmonic on \( B_{r'}(x_0) \) with \( r' > r \), then the Harnack Inequality holds on \( B_r(x_0) \), i.e. there is a \( C > 0 \) such that

\[ \sup_{x \in B_r(x_0)} u(x) < C \inf_{x \in B_r(x_0)} u(x). \]

To see this, use the previous pair of inequalities on the ball \( B_{r'}(x_0) \), where we have

\[ \sup_{x \in B_r(x_0)} u(x) \leq \sup_{x \in B_r(x_0)} \frac{(r')^{n-2} (r' + |\xi|)}{(r' - |\xi|)^{n-1}} u(0), \quad \inf_{x \in B_r(x_0)} u(x) \geq \inf_{x \in B_r(x_0)} \frac{(r')^{n-2} (r' - |\xi|)}{(r' + |\xi|)^{n-1}} u(0), \]

and thus

\[ \sup_{x \in B_r(x_0)} \frac{u(x)}{\inf_{x \in B_r(x_0)} u(x)} \leq \sup_{x \in B_r(x_0)} \frac{(r' + |\xi|)^n}{(r' - |\xi|)^n} \leq \left( \frac{2r}{r - r'} \right)^n, \]

and this is finite. Choose any \( C \) larger than this and we are done.
Now, let us assume that we have a bounded domain \( \Omega \) on which \( u \) is harmonic and \( \Omega_1 \) a smaller domain so that \( \overline{\Omega_1} \subseteq \Omega \). We want to choose an open cover of \( \Omega_1 \) using open balls that have the property that a slightly larger open ball is contained inside \( \Omega \). One way to do this is choose an open ball for every point on the boundary of \( \Omega_1 \) that is contained in \( \Omega \), then make this ball a bit smaller, and it will still contain the boundary point of \( \Omega_1 \).

Since \( \Omega_1 \) is bounded, its closure is compact, and therefore it has a finite subcover. Thus we have a finite set of balls \( B_i := B_{\delta_i}(x_i) \), \( i = 1, \ldots, m \) with the property that these balls cover \( \Omega_1 \) and each ball can be slightly enlarged and still be contained inside \( \Omega \). From this we know that the Harnack Inequality holds on each ball individually, i.e. for \( i = 1, \ldots, m \), we have

\[
\sup_{x \in B_i} u(x) < C_i \inf_{x \in B_i} u(x).
\]

Note that \( C_i \geq 1 \) by definition. Now let us consider two intersecting balls, i.e. \( B_j, B_k \) and \( B_j \cap B_k \neq \emptyset \). We then can show that

\[
\sup_{x \in B_j \cup B_k} u(x) < C_j C_k \inf_{x \in B_j \cup B_k} u(x).
\]

To see this: let \( x_0 \in B_j \) be the point at which the infimum occurs. We then know that

\[
\sup_{x \in B_j} u(x) < C_j u(x_0).
\]

Now, take any point \( z \in B_j \cap B_k \). We have that

\[
u(z) < C_j u(x_0)
\]

by the previous line, and \( u(z) \geq \inf_{x \in B_k} u(x) \), so that we have

\[
\sup_{x \in B_k} u(x) < C_k \inf_{x \in B_k} u(x) \leq C_k u(z) < C_j C_k u(x_0).
\]

Now take any two points \( y, z \in \Omega_1 \). Since we have the finite cover, there exists a path from \( y \) to \( z \) which passes through finitely many intersecting balls. Iterating the argument above gives us that there is a constant \( C \) with

\[
C < \prod_{i=1}^{m} C_i^2
\]

such that \( u(y) < C u(z) \). Since \( y, z \) were arbitrary, we are done.

4. Here we show that the Laplacian operator is also radially symmetric, or, if we consider any unitary transformation of \( \mathbb{R}^n \), the Laplacian operator remains unchanged. First, we say that an \( n \times n \) matrix \( A \) is unitary if \( AA^t = I \).

Note: Changed the notation here, it originally denoted the matrix as \( U \) instead of \( A \) but this looked a bit confusing since we also had a \( u \) for an unknown function.

(a) Show that if \( A \) is unitary, then \( \|Ax\| = \|x\| \) for all \( x \in \mathbb{R}^n \).
(b) Let $y = Ax$, where $U$ is unitary. Define
\[ \Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \quad \Delta_y u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_i^2}. \]
Show that for all $u \in C^2(\mathbb{R}^n)$, $\Delta_x u = \Delta_y u$.

Solution:

(a) We have
\[ \|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A^t Ax \rangle = \langle x, x \rangle = \|x\|^2. \]

(b) If $y = Ax$, then we have $y_i = \sum_{j=1}^{n} A_{ij} x_j$. This means that
\[ \frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} = \sum_{i=1}^{n} A_{ij} \frac{\partial}{\partial y_i}. \]
Continuing, we have
\[ \frac{\partial^2}{\partial x_j^2} = \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{n} A_{ij} \frac{\partial}{\partial y_i} \right) = \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_j} \frac{\partial}{\partial y_k} \left( \sum_{i=1}^{n} A_{ij} \frac{\partial}{\partial y_i} \right) = \sum_{i,k=1}^{n} A_{ij} A_{kj} \frac{\partial^2}{\partial y_i \partial y_k}. \]
Noting that
\[ \delta_{ik} = I_{ik} = (AA^t)_{ik} = \sum_{j=1}^{n} A_{ij}(A^t)_{jk} = \sum_{j=1}^{n} A_{ij} (A)_{kj}, \]
we have
\[ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \sum_{i,j,k=1}^{n} A_{ij} A_{kj} \frac{\partial^2}{\partial y_i \partial y_k} = \sum_{i,k=1}^{n} \delta_{ik} \frac{\partial^2}{\partial y_i \partial y_k} = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}. \]

5. (McOwen 4.4.2).

Solution: We follow the procedure of separation of variables from earlier. If we write
\[ u(x, y, z) = A(x)B(y)C(z), \]
then we obtain the system
\[ \frac{\alpha''(x)}{A(x)} = \frac{\beta''(y)}{B(y)} = \frac{\gamma''(z)}{C(z)}, \]
and these must all be constant. We again have boundary conditions $A(0) = A(a) = 0$, etc. so that our eigenfunctions are
\[ u_{k\ell m}(x, y, z) = \sin \left( \frac{k \pi x}{a} \right) \sin \left( \frac{l \pi y}{b} \right) \sin \left( \frac{m \pi z}{c} \right), \]
and this has eigenvalue
\[ \lambda_{k\ell m} = \pi^2 \left( \frac{k^2}{a^2} + \frac{l^2}{b^2} + \frac{m^2}{c^2} \right). \]
Solution:

(a) If we write \( y(r) = J_n(\sqrt{\lambda} r) \), then
\[
y'(r) = \sqrt{\lambda} J_n'(\sqrt{\lambda} r), \quad y''(r) = \lambda J_n''(\sqrt{\lambda} r),
\]
and
\[
r^2 y'' + ry' + (\lambda r^2 - n^2) y = r^2 \lambda J_n''(\sqrt{\lambda} r) + r \sqrt{\lambda} J_n'(\sqrt{\lambda} r) + (\lambda r^2 - n^2) J_n(\sqrt{\lambda} r).
\]
Writing \( s = \sqrt{\lambda} r \) gives
\[
s^2 J_n''(s) + s J_n'(s) + (s^2 - n^2) J_n(s) = 0,
\]
by the definition of \( J_n \).

(b) In rectangular coordinates, \( \Delta u = u_{xx} + u_{yy} \). Switching to polar coordinates through the transformation
\[
x = r \cos \theta, \quad y = r \sin \theta
\]
and applying the chain rule, we have
\[
\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.
\]
If we write \( u(r, \theta) = A(r) B(\theta) \), then the eigenvalue equation becomes
\[
A''(r) B(\theta) + \frac{1}{r} A'(r) B(\theta) + \frac{1}{r^2} A(r) B''(\theta) + \lambda A(r) B(\theta) = 0.
\]
If we divide everything by \( A(r) B(\theta) \) and multiply by \( r^2 \), we obtain
\[
r^2 \frac{A''(r)}{A(r)} + r \frac{A'(r)}{A(r)} + \lambda r^2 = -\frac{B''(\theta)}{B(\theta)}.
\]
Since the LHS is independent of \( \theta \) and the RHS is independent of \( r \), they must both be constant, so set each equal to \( \gamma^2 \). This gives
\[
B''(\theta) = -\gamma^2 B(\theta), \quad B(\theta + 2\pi) = B(\theta).
\]
The ODE has the solution
\[
B(\theta) = \alpha \cos(\gamma \theta) + \beta \sin(\gamma \theta),
\]
and the boundary condition requires that \( \gamma \in \mathbb{N} \). We we have
\[
B_n(\theta) = \alpha_n \cos(n \theta) + \beta_n \sin(n \theta).
\]
This then gives
\[
r^2 \frac{A''(r)}{A(r)} + r \frac{A'(r)}{A(r)} + \lambda r^2 = n^2,
\]
which becomes
\[ r^2 A''(r) + rA'(r) + (\lambda r^2 - n^2)A(r) = 0, \quad A(a) = 0. \]

The ODE is the Bessel ODE from above, so we see that \( J_n(\sqrt{\lambda}r) \) is a solution. Moreover, the boundary conditions gives that \( \sqrt{\lambda}a = \rho_{n,k} \) for some \( k \geq 1 \) (noting that \( \rho_{n,k} \) are the zeros of \( J_n(\cdot) \)). Therefore this has a solution iff \( \lambda = \lambda_{n,k} \), where
\[ \lambda_{n,k} = \rho_{n,k}^2/a^2. \]

Therefore, our eigenfunctions and eigenvalues are
\[ u_{n,k}(r, \theta) = J_n(\rho_{n,k}r/a)(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)), \quad \lambda_{n,k} = \rho_{n,k}^2/a^2. \]

(c) We have
\[ \Delta u_{n,k} + \lambda_{n,k} u_{n,k} = 0. \]

Writing
\[ u(r, \theta, t) = \sum_{n,k} \alpha_{n,k}(t)u_{n,k}(r, \theta), \]

and plugging into the wave equation, we obtain the family of equations
\[ \alpha_{n,k}''(t) = -\lambda_{n,k}\alpha_{n,k}(t), \]
giving
\[ \alpha_{n,k}(t) = A_{n,k} \cos(\rho_{n,k}t/a) + B_{n,k} \sin(\rho_{n,k}t/a). \]

Plugging in \( t = 0 \) gives
\[ g(r, \theta) = \sum_{n,k} A_{n,k}u_{n,k}(r, \theta), \]
so we can obtain the \( A_{n,k} \) from the series expansion of \( g \), and differentiating in time and plugging in \( t = 0 \) gives
\[ h(r, \theta) = \sum_{n,k} a^{-1} B_{n,k} \rho_{n,k} u_{n,k}, \]
and we can obtain the \( B_{n,k} \) from the series expansion of \( h \).