1. (McOwen 4.1.6). Show that \( v(x) = |x - x_0|^2 - n \) is harmonic in \( \mathbb{R}^n \setminus \{x_0\} \) for all \( n \geq 3 \). If \( n = 2 \), show that \( v(x) = \log |x - x_0| \) is harmonic in \( \mathbb{R}^2 \setminus \{x_0\} \).

2. (McOwen 4.1.5). Let \( q(x) \geq 0 \) for \( x \in \Omega \) and consider solutions \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) of the PDE
\[
\Delta u - q(x)u = 0 \quad \text{in} \quad \Omega.
\]
Establish uniqueness theorems for both the Dirichlet and Neumann problems.

3. Let \( u(x) \) be a harmonic function on \( \mathbb{R}^n \setminus \{0\} \) that is radially symmetric, i.e. there is a function \( \psi: \mathbb{R}^+ \rightarrow \mathbb{R} \) such that \( u(x) = \psi(|x|) = \psi(r) \). Show that
\[
\psi''(r) + \frac{n-1}{r}\psi'(r) = 0.
\]

4. Consider Laplace’s equation on a square with Dirichlet BC, i.e. \( \Omega = (0, \pi) \times (0, \pi) \), and
\[
\Delta u = 0, \quad x \in \Omega,
\]
\[
u(x,0) = a(x), \quad u(x,\pi) = b(x),
\]
\[
u(0,y) = c(y), \quad u(\pi,y) = d(y),
\]
where we assume that all four functions \( a, b, c, d \) are zero at \( 0, \pi \). Write down a series solution for this PDE.

**Hint:** We did the case where \( a = c = d = 0 \) in class. First write down the three other cases where we choose only one of the boundary functions to be nonzero, then use linearity.

5. Let \( \Omega = (-1,1) \times (-1,1) \) and consider the problem
\[
\Delta u = 0, \quad x \in \Omega,
\]
\[
u(-1,y) = u(1,y) = 0,
\]
\[
u(x,-1) = u(x,1) = f(x),
\]
where
\[
\begin{align*}
&\bullet \ f \ \text{is even} \\
&\bullet \ f(x) \ \text{is strictly decreasing on} \ [0,1] \\
&\bullet \ f(\pm1) = 0
\end{align*}
\]
Show that \( u \) has a saddle point at \( (0,0) \) and \( u(0,0) > 0 \).