Math 553. Homework 2. Solutions.

1. In class we referred to Leibnitz’s Rule, namely:

\[
\frac{d}{dy} \int_a^{\xi(y)} u(x, y) \, dx = \xi'(y) u(\xi(y) -, y) + \int_a^{\xi(y)} \frac{\partial u}{\partial y}(x, y) \, dx,
\]

where

\[ u(z-, y) := \lim_{x \to z-} u(x, y). \]

Prove this formula holds whenever \( u(x, y) \) is \( C^1 \) with domain \((a, \xi(y)) \times \mathbb{R}\). What happens if the \( \xi(y) \) is in the lower limit of integration?

Solution: Let us write

\[ G(y) = \int_a^{\xi(y)} u(x, y) \, dx, \]

and we want to compute \( G'(y) \). Proceeding formally, we have

\[ G(y + h) = \int_a^{\xi(y+h)} u(x, y + h) \, dx, \]

and

\[ G(y + h) - G(y) = \int_a^{\xi(y+h)} u(x, y + h) \, dx - \int_a^{\xi(y)} u(x, y) \, dx = \int_a^{\xi(y)} (u(x, y + h) - u(x, y)) \, dx + \int_{\xi(y)}^{\xi(y+h)} u(x, y + h) \, dx. \]

If \( u, \xi \in C_1 \) then

\[ u(x, y + h) = u(x, y) + h \frac{\partial u}{\partial y}(x, y) + O(h^2), \quad \xi(y + h) = \xi(y) + h\xi'(y) + O(h^2). \]

Also notice that since \( u \in C^1 \), then

\[ \max_{x \in (a, a+O(h))} u(x, y) = u(a-, y) + O(h). \]

Writing

\[ \min_{x \in (\xi(y), \xi(y+h))} u(x, y) = m, \quad \max_{x \in (\xi(y), \xi(y+h))} u(x, y) = M, \]

we have

\[ M = u(\xi(y) -, y) + O(h), \quad m = u(\xi(y) -, y) - O(h). \]

We then have

\[ (\xi(y + h) - \xi(y))m \leq \int_{\xi(y)}^{\xi(y+h)} u(x, y + h) \, dx \leq (\xi(y + h) - \xi(y))M. \]
Putting this all together, we have
\[
\lim_{h \to 0^+} \frac{G(y + h) - G(y)}{h} = \lim_{h \to 0^+} \int_a^\xi(y) \frac{u(x, y + h) - u(x, y)}{h} \, dx + \lim_{h \to 0^+} \frac{1}{h} \int_{\xi(y)}^{\xi(y + h)} u(x, y + h) \, dx
\]
\[
= \int_a^\xi(y) \frac{\partial u}{\partial y}(x, y) \, dx + \lim_{h \to 0^+} \frac{1}{h}(\xi(y + h) - \xi(y))(u(\xi(y) -, y) + O(h))
\]
\[
= \int_a^\xi(y) \frac{\partial u}{\partial y}(x, y) \, dx + \xi'(y)(u(\xi(y) -, y).
\]

2. Consider again Burgers
\[
uu_x + u_y = 0, \quad u(x, 0) = h(x),
\]
where we assume that \( h \in C^1 \) and \( 0 < h'(x) < A < \infty \).

(a) Show that this PDE has a global solution on \( y > 0 \), and that no shocks or rarefactions develop.

(b) Write down as explicit a formula as you can for \( u(x, y) \).

(c) Explain using physical arguments why this initial condition gives global existence whereas other initial conditions do not.

Solution:

(a) Recall from the derivation of characteristics for Burger’s, we have
\[
x' = z, \quad y' = 1, \quad z' = 0.
\]
This means that the characteristic starting at \((x_0, 0)\) can be parameterized by
\[
x(t) = h(x_0)t + x_0, \quad y(t) = t.
\]
If we can show that each point in the upper half plane is on a unique characteristic, then we are done. But solving
\[
(h(x_0)t + x_0, t) = (x, y)
\]
gives the following problem: given \((x, y)\), does there exist a unique \( x_0 \) so that
\[
H_y(x_0) := h(x_0)y + x_0 = x?
\]
Another way of asking this is: is the function \( H_y: \mathbb{R} \to \mathbb{R} \) invertible? We see that
\[
H_y'(x_0) = h'(x_0)y + 1 > 1
\]
by the assumptions of the problem. This means that \( H_y \) is invertible: increasing since it always has a positive derivative, and in fact onto because its derivative has a positive lower bound.
(b) Using the computation above: if we know the value of $x_0$ such that $H_y(x_0) = x$, then

$$u(x, y) = h(x_0).$$

This means that

$$u(x, y)y + h^{-1}(u(x, y)) = x, \quad x - u(x, y)y = h^{-1}(u(x, y)).$$

In general we cannot go much further than this. There are a few special cases that we can work out, however. For example, if we choose $h(x) = \alpha x$, then we must solve

$$\alpha y z + z = x, \quad z = \frac{x}{1 + \alpha y},$$

so $u(x, y) = \alpha x / (1 + \alpha y)$. We can plug in, noting that

$$u_x = \frac{\alpha}{1 + \alpha y}, \quad u_y = \frac{-\alpha^2 x}{(1 + \alpha y)^2},$$

and so $u_y = -uu_x$.

A more complicated example would be if we choose

$$h(x) = \begin{cases} x, & x < 0, \\ 2x, & x > 0. \end{cases}$$

We then have

$$H_y(z) = \begin{cases} (y + 1)z, & x < 0, \\ (2y + 1)z, & x > 0. \end{cases}$$

Solving $H_y(z) = x$ gives

$$z = \begin{cases} \frac{x}{y + 1}, & x < 0, \\ \frac{2x}{2y + 1}, & x > 0. \end{cases}$$

Finally, we have

$$u(x, y) = h(z) = \begin{cases} \frac{x}{y + 1}, & x < 0, \\ \frac{2x}{2y + 1}, & x > 0. \end{cases}$$

(c) Consider the fundamental model of a stream of particles moving with a fixed velocity. Our assumption on the initial data means that particles that start to the right move faster, so we have no danger of particles colliding. Moreover, since the initial data is smooth, we do not expect a rarefaction wave, since the density of particles should stay continuous as well.

3. Consider the traffic flow equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

where we assume

$$q = G(\rho) = c\rho(1 - \rho/\rho_{\text{max}}).$$

(a) Give an interpretation of the constants $c$ and $\rho_{\text{max}}$. 

(b) Suppose that the initial condition is given by

\[ \rho(x, 0) = \begin{cases} \rho_{\text{max}}, & x < 0, \\ \frac{\rho_{\text{max}}}{2}, & x > 0. \end{cases} \]

Show that this system has a shock, determine the position of the shock for all \( t > 0 \), and determine the density \( \rho(x, t) \). What is the interpretation of this solution?

(c) Consider the case where there has been some stoppage (e.g. accident, obstruction) in the road for some time, and at time \( t = 0 \) the obstruction is cleared. This initial condition can be described as

\[ \rho(x, 0) = \begin{cases} \rho_{\text{max}}, & x < 0, \\ 0, & x > 0. \end{cases} \]

Describe this solution as completely as possible.

(d) In each of the previous two cases, imagine that you are in a car located at \( x = -1 \) at time \( t = 0 \). Describe your position as a function of time.

**Note:** Be careful when determining the characteristics for this problem; in particular, recall that while the slope of the characteristics for Burgers equation are equal to the value of the function, this is not true here!

---

**Solution:**

(For the record, I was expecting the solutions to parts (a)—(c) to be more or less straightforward, but (d) might have been particularly difficult.)

(a) We call that \( q \) is the traffic flow of cars per minute. When \( \rho \) is small, we see that \( G(\rho) \approx c\rho \), meaning that the flow of cars is \( c \) times the density. This means that the typical velocity when the density is small is \( c \). Think of \( c \) as the (effective) speed limit of how fast a single car will drive on a completely empty highway.

Conversely, \( \rho_{\text{max}} \) is the density at which there is no flow, i.e. there are no cars moving. Think of this as bumper-to-bumper traffic where no car can move.

Notice also that the maximum value of \( G(\rho) \) is when \( \rho = \rho_{\text{max}}/2 \), and then \( G(\rho_{\text{max}}/2) = c\rho_{\text{max}}/2(1 - 1/2) = c\rho_{\text{max}}/4 \).

(b) Let us first map out the characteristics. Notice that we can write this equation in the form

\[ u_y + G'(\rho)\rho_x = 0 \]

(we are writing time as \( y \) just to simplify the graphical understanding). If we move along the characteristic \( x(t), y(t) \), writing \( z(t) = u(x(t), y(t)) \), then we obtain

\[ x' = G'(z), \quad y' = 1, \quad z' = 0. \]

So the solution is constant along characteristics (\( z' = 0 \)), and the characteristics are straight lines. But notice that their \( x \) derivative is not \( z \) (as it is in Burger’s) but

\[ G'(z) = c - \frac{2c}{\rho_{\text{max}}} z. \]
When \( z = \rho_{\text{max}}/2 \), we have \( G'(z) = 0 \), and when \( z = \rho_{\text{max}} \), we have \( G'(z) = -c \). Therefore the characteristics are vertical in the left-hand half plane, but slant to the left in the right-hand half plane. So we expect a shock at zero to travel to the left.

Using Rankine–Hugoniot, if \( \xi(y) \) is the location of the shock for \( y > 0 \), then

\[
\xi'(y) = \frac{G(u_R) - G(u_L)}{u_R - u_L}.
\]

In this case, we have \( u_L = \rho_{\text{max}}/2 \) and \( u_R = \rho_{\text{max}} \), giving

\[
\xi'(y) = \frac{0 - c\rho_{\text{max}}/4}{\rho_{\text{max}} - \rho_{\text{max}}/2} = -\frac{c}{2}.
\]

Since \( \xi(0) = 0 \), we have \( \xi(y) = -cy/2 \). This gives

\[
\rho(x, y) = \begin{cases} 
\rho_{\text{max}}/2, & x < -cy/2, \\
\rho_{\text{max}}, & x > -cy/2.
\end{cases}
\]

The physical interpretation of this is as follows: at time \( t = 0 \), there is a huge traffic jam to the right of \( x = 0 \). All of the cars to the left of zero are moving to the right with high velocity. As they approach the traffic jam, they need to slow down to velocity zero to not crash. This traffic jam is then piling up, and will effectively move to the left as time increases. Also notice that the speed of the shock is roughly half of the speed limit, which is the speed at which cars are entering the shock (see part (d) for more on this).

(c) In this case, we obtain the same information about characteristics. Characteristics that start to the left of the origin move to the left with velocity \(-c\), as we computed before, and characteristics to the right of the origin move with velocity \( G'(0) = c \). So we have a case of an underdetermined solution, with a fan spreading from the origin whose left edge is the line \( y = -x/c \) and whose right edge is \( y = x/c \). Therefore we know

\[
\rho(x, y) = \begin{cases} 
0, & x > cy, \\
\rho_{\text{max}}, & x < -cy, \\
? & -cy < x < cy.
\end{cases}
\]

Assuming that we have a rarefaction wave (and not some artificially introduced shock), we should use linear extrapolation. A linear function which goes from \( \rho_{\text{max}} \) at \(-cy\) to 0 at \( cy \) needs to have a slope of \(-\rho_{\text{max}}/2cy\), so we can replace the ? with

\[
\rho(x, y) = \begin{cases} 
0, & x > cy, \\
\frac{\rho_{\text{max}}}{2} \left(1 - \frac{x}{cy}\right), & -cy < x < cy, \\
\rho_{\text{max}}, & x < -cy.
\end{cases}
\]

(d) Let us now imagine that we are a driver starting at \( x = -1 \) at time \( y = 0 \). Notice that since the flow rate is \( q = G(\rho) \) with density \( \rho \), the velocity of a car given the density is given by \( G(\rho)/\rho = c(1 - \rho/\rho_{\text{max}}) \). Notice that the maximum speed at any density is given by the low-density limit, so no car will travel faster than \( c \).
First consider case (b). In this case, there is a left-moving shock, to the left of which there is motion and to the right of which, there is gridlock. So the only thing to figure out is when we hit the shock, and stop. Also, since we will be moving to the right, we are guaranteed to hit the shock. The only question remains as to when we will hit the shock. Our velocity until we hit the shock is \(\frac{c}{2}\), and the shock is moving to the left at speed \(-\frac{c}{2}\), so the rate at which we approach the shock is \(c\). Therefore we will hit the shock at time \(\frac{1}{c}\), when we are located at \(x = -\frac{1}{2}\) and at time \(y = \frac{1}{c}\). Thus we get “halfway there”. (In fact, the “halfway there” calculation holds for all drivers, since they are moving at the same velocity as the shock.) In this case, we can parameterize our position as

\[
p(t) = \begin{cases} 
-1 + \frac{c}{2}t, & t < \frac{1}{c} \\
-\frac{1}{2}, & t \geq \frac{1}{c} 
\end{cases}
\]

Now consider (c). In this case, we will be motionless until the left-hand side of the rarefaction fan hits us, at which time we start moving. Note that no car can ever exit the rarefaction fan once it enters; there are two ways to see this: the first is to note that density to the right is zero, so no cars there, but also we can see that the right-hand edge of the fan is moving at velocity \(c\), and no car can move faster than \(c\), so it is impossible to catch up. This means that once we are in the fan, we stay there forever, so we can just use the middle term of the solution. If \(p(t)\) is the position of our car at time \(t\), then the velocity of our car is \(G(\rho)/\rho\), where \(\rho\) is the density at the point \((p(t), t)\). Thus we have

\[
\frac{dp}{dt} = c(1 - \frac{\rho(p,t)}{\rho_{\text{max}}}) = c\left(1 - \frac{1}{2}\left(1 - \frac{p}{ct}\right)\right) = \frac{c}{2} - \frac{p}{2t}.
\]

Also notice that we have the initial condition \(p(1/c) = -1\), because until time \(1/c\) we are unable to move. We can solve this (using, for example, integrating factors) and obtain

\[
p(t) = ct - 2\sqrt{ct}, \quad t > \frac{1}{c}.
\]

So we are always inside the fan; we are getting further away from the front of the pack in space, but notice that our average velocity over our whole trip is approaching \(c\) from below. We can also see then that

\[
\rho(p(t), t) = \frac{\rho_{\text{max}}}{\sqrt{ct}}
\]

so we are in the low-traffic regime: in fact the density of cars around us go to zero. In fact, for any initial position \(x_0\), we see something similar: if we assume that our starting location is \(x_0 < 0\), then the left edge of the fan will hit us at time \(-x_0/c\), so we solve the ODE with IC \(p(-x_0/c) = x_0\) which then gives

\[
p(t) = ct - 2\sqrt{|x_0|ct}.
\]

In fact the asymptotics scale the same no matter where we start, but the further we start to the left, the further we will be from the front of the pack for all time.
4. Consider Burgers equation with initial data that gives rise to a rarefaction wave, namely:

\[ u_y + uu_x = 0, \quad u(x, 0) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0 \end{cases} \]

Choose \( n > 1 \).

(a) Show that there is a solution that takes on exactly the \( n + 1 \) values \( \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\} \) and has exactly \( n \) shocks. Compute the location of each shock.

(b) Write down the formula for your solution in the form

\[ u(x, y) = \ldots \]

(c) Show directly that your formula is a weak solution.

Solution:

(a) Recall that computing details of a shock is always a local calculation, so we can consider each shock separately (assuming they don’t collide). Let us first compute the speed of a shock with value \( k/n \) to the left and \( (k + 1)/n \) to the right. Recall Rankin–Hugoniot:

\[ \xi'(t) = \frac{G(u_L) - G(u_R)}{u_L - u_R} = \frac{u_L^2/2 - u_R^2/2}{u_L - u_R} = \frac{1}{2}(u_L + u_R), \]

so for this shock we have

\[ \xi'(t) = \frac{1}{2} \left( \frac{k}{n} + \frac{k + 1}{n} \right) = \frac{2k + 1}{2n}. \]

For example, the speed of the shock between 0 and 1/n will move at speed 1/2n, the one between 1/n and 2/n will move at speed 3/2n, etc. Note that for larger \( k \), the shock velocity is larger, so that shocks will not crash into each other.

We will call the shock between 0 and 1/n the 1st shock, and between 1/n, 2/n the second shock, etc. so that the number of the shock is the numerator of the right-hand value.

Also, using the equation, we have that the location of the \( k + 1 \)st shock at time \( t \) is just

\[ (x, t) = \left( \frac{2k + 1}{2n} t, t \right), \]

or, if we are plotting \( t \) as a function of \( x \) (if we represent the vertical axis as \( t \), for example), we have the line

\[ t = \frac{2n}{2k + 1} x. \]

Again, larger \( k \) means smaller slope, so again we see that these solutions do not crash into each other.
(b) Using the previous analysis, we have that \((x, t)\) is to the left of the 1st shock, then \(u = 0\), if \((x, t)\) is between the 1st and 2nd shocks, we have \(u = 1/n\), etc. More generally, if \((x, t)\) is between the \(k\)th and \(k+1\)st shocks, then \(u = k/n\). Thus we have

\[
  u(x, t) = \begin{cases} 
  0, & x < t/2n, \\
  1, & x > (2n + 1)/(2n)t.
\end{cases}
\]

(c) 5. Consider the PDE

\[ u_t = u_{xxx}, \quad u(x, 0) = \sin(x). \]

Write out a power series expansion for this solution under the assumption that it is analytic.

Solution: We first need to determine all of the derivatives of \(u(x, t)\) at some point. For simplicity we can choose \((0, 0)\), although in fact we could do it at any \((x_0, 0)\).

Let us write \(f(x) = \sin(x)\). Note that

\[
  c_n := f^{(n)}(0) = \begin{cases} 
  1, & n = 1 \mod 4, \\
  -1, & n = 3 \mod 4, \\
  0, & n \text{ even}.
\end{cases}
\]

We have

\[
  u_{x^n}(0, 0) = c_n.
\]

Since \(u_t = u_{xxx}\), we have

\[
  u_{tx^n} = (u_t)_{x^n} = (u_{xxx})_{x^n} = u_{x^{n+3}},
\]

so

\[
  u_{tx^n}(0, 0) = c_{n+3}.
\]

Similarly, for any \(a, b \in \mathbb{N}\), we have

\[
  u_{t^a x^b} = c_{3a+b}.
\]

We could put the final answer in the form

\[
  u(x, t) = \sum_{a, b=0}^{\infty} \frac{c_{3a+b}}{a!b!} t^a x^b,
\]

but this can also be simplified. Let us figure out the modulo 4 pattern for \(3a + b\), which we can see in the following table:

<table>
<thead>
<tr>
<th>(b) | (a)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\(3a + b = 3a + (a + 1) = 4a + 1 = 1 \mod 4\).
Similarly, we have $3a + b = 3 \mod 4$ if $b = a - 1$, since $3a + b = 4a - 1 = -1 \mod 4 = 3 \mod 4$.

Finally, we see that if $a, b$ have the same parity (either both even or both odd), then $3a + b$ is even.

Then we can write our power series as

$$
\sum_{a=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^a x^{a+4k+1}}{a! (a+4k+1)!} - \sum_{a=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^a x^{a+4k-1}}{a! (a+4k-1)!} = \sum_{a=0}^{\infty} \frac{t^a x^a}{a!} \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k-1}}{(4k-1)!}.
$$

We can in fact manipulate this further to get a much nicer solution, but stopping here is fine as well.