1. Our classroom is Everitt 168. The integer 168 has many interesting properties. Describe three distinct such properties.

Solution. There are many interesting properties at

https://en.wikipedia.org/wiki/168_(number)

Some of my personal favorites are

- size of the group $GL_3(\mathbb{F}_2)$, the automorphism group of the original Hamming code
- number of hours in a week
- number of dots on a set of dominoes

2. Consider the heat equation on the real line, i.e.

$$u_t = ku_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$  

Consider

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{x^2}{4kt} \right).$$

(a) Show that $u(x, t)$ is a solution to the heat equation that is well-defined for all $t > 0$.
(b) Describe what this solution looks like as $t \to 0+$.
(c) Compute

$$\int_{-\infty}^{\infty} u(x, t) \, dx.$$  

d) Let $-\infty < a < b < \infty$. Compute

$$\lim_{t \to \infty} \int_a^b u(x, t) \, dx.$$  

(e) Describe in words what the last two computations tell us.

Solution.

(a) We compute:

$$\frac{\partial}{\partial t} u(x, t) = k \frac{e^{\frac{x^2}{4kt}} (-2kt + x^2)}{8\sqrt{\pi} (kt)^{5/2}},$$  

and

$$\frac{\partial^2}{\partial x^2} u(x, t) = \frac{e^{\frac{x^2}{4kt}} (-2kt + x^2)}{8\sqrt{\pi} (kt)^{5/2}},$$

so this satisfies the heat equation. Also, we see that both of these are continuous for all $x$ and all $t > 0$.  


(b) First note that $u(0,t) = (4\pi kt)^{-1/2}$, so

$$\lim_{t \to 0^+} u(0,t) = \infty.$$  

Now, if $x \neq 0$, we have a prefactor that blows up like a square root, but an exponential term that decays exponentially fast. So the product should decay to zero. To check this more carefully, let us write $s = \sqrt{4kt}$, and up to a constant factor we want to compute

$$\lim_{s \to 0^+} \frac{1}{s} e^{-x^2/s^2} = \lim_{s \to 0^+} \frac{1}{s} e^{-x^2/s^2}.$$

(Note that $t \to 0^+$ iff $s \to 0^+$ so this limit is the same.) We see that the last fraction is an indeterminate $\infty/\infty$ form, so we apply l’Hôpital’s Rule:

$$\lim_{s \to 0^+} \frac{1}{s} e^{-x^2/s^2} = \lim_{s \to 0^+} \frac{-1/s^2}{e^{-x^2/s^2}(-2x^2/s^3)} = \lim_{s \to 0^+} \frac{s/2}{e^{-x^2/s^2}} = 0/\infty = 0.$$

Collecting this gives

$$\lim_{t \to 0^+} u(x,t) = \begin{cases} \infty, & x = 0, \\
0, & x \neq 0.\end{cases}$$

Qualitatively, what is happening is that we have a Gaussian curve which is getting more narrow but also growing at the origin.

(c) If we make the change of variables $y = x/\sqrt{4kt}$, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{x^2}{4kt} \right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

Thus the total mass of the system is preserved and is constantly one.

(d) We have

$$\int_{a}^{b} u(x,t) \, dx = \left( \int_{-\infty}^{\infty} - \int_{b}^{\infty} - \int_{-\infty}^{a} \right) u(x,t) \, dx.$$

We have already computed the first integral to be 1. Let us consider the second integral,

$$R(b) = \int_{b}^{\infty} \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{x^2}{4kt} \right) dx.$$

Again let us change variables $y = x/\sqrt{4kt}$ and obtain

$$\lim_{t \to \infty} \int_{b}^{\infty} \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{x^2}{4kt} \right) dx = \lim_{t \to \infty} \int_{b/\sqrt{4kt}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}.$$ 

Similarly, we have

$$\lim_{t \to \infty} \int_{-\infty}^{a} u(x,t) \, dx = 1/2,$$

so that

$$\int_{a}^{b} u(x,t) \, dx \to 0.$$
(e) What this says is that while the mass is conserved on the line, all of the mass escapes any finite set, so it “leaks” toward infinity. (In fact, we have shown something slightly stronger in our proof method, namely that exactly half escapes to $+\infty$ and half to $-\infty$.)

3. (McOwen 1.1.2). If $S_1, S_2$ are two graphs (i.e. $S_i$ is given by $z = u_i(x, y)$) that are integral surfaces of $V = \langle a, b, c \rangle$ and intersect in the curve $\chi$, show that $\chi$ is a characteristic curve.

4. (McOwen 1.1.4). Solve the given initial value problem and determine the values of $x, y$ for which it exists:

(a) $x u_x + u_y = u, \quad u(x, 0) = x^2$
(b) $u_x - 2u_y = u, \quad u(0, y) = y$
(c) $y^{-1} u_x + u_y = u^2, \quad u(x, 1) = x^2$

Solution.

(a) The characteristic equations can be seen to be

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = z.$$

We parametrize the initial data as $\Gamma = (s, 0, s^2)$. Note that the tangent vector to $\Gamma$ is $(1, 0, 2s)$ which can never be a scalar multiple of $(x, 1, z)$, so that the initial data is noncharacteristic and we can write down a unique solution. Appending the initial conditions $x(0) = s, \quad y(0) = 0, \quad z(0) = s^2$,

we solve and obtain

$$x(s, t) = se^t, \quad y(s, t) = t, \quad z(s, t) = s^2 e^t.$$

We can invert these to obtain

$$s(x, y) = xe^{-y}, \quad t(x, y) = y,$$

giving

$$u(x, y) = z(s(x, y), t(x, y)) = (xe^{-y})^2 e^y = x^2 e^{-y}.$$

(b) The characteristic equations can be seen to be

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2, \quad \frac{dz}{dt} = z.$$

We parametrize the initial data as $\Gamma = (0, s, s).$ Note that the tangent vector to $\Gamma$ is $(0, 1, 1)$ which can never be a scalar multiple of $(1, -2, z)$, so that the initial data is noncharacteristic and we can write down a unique solution. Appending the initial conditions $x(0) = 0, \quad y(0) = s, \quad z(0) = s$,

we solve and obtain

$$x(s, t) = t, \quad y(s, t) = s - 2t, \quad z(s, t) = se^t.$$
We can invert these to obtain
\[ s(x, y) = 2x + y, \quad t(x, y) = x, \]
giving
\[ u(x, y) = z(s(x, y), t(x, y)) = (2x + y)e^x. \]

(c) The characteristic equations can be seen to be
\[ \frac{dx}{dt} = \frac{1}{y}, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = z^2. \]
We parametrize the initial data as \( \Gamma = (s, 1, s^2) \). Note that the tangent vector to \( \Gamma \) is \((1, 0, 2s)\) which can never be a scalar multiple of \((1/y, 1, z^2)\), so that the initial data is noncharacteristic and we can write down a unique solution. Appending the initial conditions
\[ x(0) = s, \quad y(0) = 1, \quad z(0) = s^2, \]
we solve and obtain
\[ y(s, t) = 1 + t. \]
Plugging this into the equation for \( x \) we obtain
\[ \frac{dx}{dt} = \frac{1}{1 + t}, \quad x(0) = s. \]
We can solve this to obtain
\[ x(t) = \log(1 + t) + s. \]
Finally, we have
\[ z(t) = \frac{z(0)}{1 - tz(0)} = \frac{s^2}{1 - ts^2}. \]
We can invert \( x, y \) to obtain
\[ s(x, y) = x - \log y, \quad t(x, y) = y - 1, \]
giving
\[ u(x, y) = z(s(x, y), t(x, y)) = \frac{(x - \log y)^2}{1 - (y - 1)(x - \log y)^2}. \]

5. In this problem, you are to construct a semilinear PDE of the form
\[ a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} = c(x, y, u), \]
whose solution has the following properties:

- The initial conditions are given by \( u(x, 0) = h(x) \);
- Let \( \Omega \) be the largest open set for which \( u(x, y) \) is defined given \( h \). \( \Omega \) should have the property that it contains the entire upper half plane and is contained in a translate of the upper half plane, i.e. there exists \( y^* < 0 \) such that
\[ \{(x, y) | y > 0\} \subseteq B \subseteq \{(x, y) | y > y^*\}. \]
You are to specify $a, b, c, h$ so that the solution has those properties.

**Solution.** One way to think of a solution for this to cook up a case where the characteristics cover the whole plane, but the ODE along each characteristic is well-behaved as $y \to \infty$ but not as $y \to -\infty$.

So, for example, one thing we could do is choose $a \equiv 1, b \equiv 1$. In this case, the characteristic curves are given by lines with slope 1 that move up and to the right, and clearly these curves foliate the plane.

Now, we need an ODE that blows up in finite backward time but not in forward time. One possible example is the quadratic ODE $z' = -z^2$, with $z(0) > 0$. We can solve this explicitly to obtain

$$z(t) = \frac{z(0)}{1 + tz(0)},$$

and we see that this has a finite-time blowup at $t = -1/z(0)$.

Thus if we travel along the characteristic that intersects the $x$-axis at $(x_0, 0)$, then the solution will be well-behaved for all $t > 0$ (which corresponds to $y > 0$), but will only exist for $t > -1/z(0) = -1/h(x_0)$. Since we want this time to be negative, we also need to choose $h(x_0) > 0$ for all $x_0$.

More formally: let us parameterize the initial curve $\Gamma$ by $(s, 0, h(s))$. Then the equations for the characteristics in this case are

$$\frac{dx}{dt} = 1, x(0) = s, \quad \frac{dy}{dt} = 1, y(0) = 0,$$

which gives

$$x(s, t) = s + t, \quad y(s, t) = t.$$

We can of course invert that to obtain $s = x - y, t = y$. The equation for $z$ is

$$\frac{dz}{dt} = -z^2, \quad z(0) = h(s),$$

giving

$$z(s, t) = \frac{h(s)}{1 + th(s)}.$$  

Then we have

$$u(x, y) = z(s(x, y), t(x, y)) = \frac{h(x - y)}{1 + yh(x - y)}.$$  

If $h > 0$, then the denominator is always non-zero for $y > 0$, so the function is defined there. But for any fixed $x$, the function has a singularity at $y = -1/h(x - y)$. So, for example, if we assume that there exist $C_1, C_2$ such that

$$0 < C_1 < h(x) < C_2$$

for all $x$,

then we can choose $1/C_1 > y^* > 1/C_2$.  