Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 4 — due October 9

1. Consider the boundary value problem

\[ A'' + \lambda A = 0, \quad A'(0) + aA(0) = 0, \quad A(L) = 0. \]

(a) Show that if \( a < 0 \), then there is no negative eigenvalue.
(b) Under which conditions is there a zero eigenvalue?
(c) Show there are infinitely many positive eigenvalues for any value of \( a \).

**Bonus:** We showed in (a) that if \( a < 0 \) then there is no negative eigenvalue. It turns out that for some positive \( a \), this problem has a negative eigenvalue (and for some others it does not). Write down a condition on \( a \) which determines whether such an eigenvalue exists.

2. *(Strauss 4.3.2.)* Consider the eigenvalue problem with Robin boundary conditions

\[ A'' + \lambda A = 0, \quad A'(0) - \alpha_0 A(0) = 0, \quad A'(L) + \alpha_L A(L) = 0. \]

(a) Show that zero is an eigenvalue if and only if \( \alpha_0 + \alpha_L = -\alpha_0 \alpha_L \).
(b) Compute the eigenfunction corresponding to this eigenvalue.

3. Solve the equation

\[ u_t = ku_{xx}, \quad x \in [0, \infty), \quad t > 0, \]

\[ u(x, 0) = \begin{cases} 1, & x \in (0, 1), \\ 0, & x > 1, \end{cases} \]

\[ u(0, t) = 0. \]

4. Consider the Schrödinger equation with Neumann boundary conditions:

\[ iu_t = u_{xx}, \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0. \]

Write out the general series solution for this equation as we have done for the heat and wave equations, i.e. separate variables, get ODEs in \( x \) and \( t \), solve these problems, and take the linear combination.