Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 2 — due September 18

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1. **(Strauss 2.1.1.)** Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

   Solution: We use d'Alembert's formula with

   $$\phi(x) = e^x, \quad \psi(x) = \sin x.$$

   Then we have

   $$\int_{x-ct}^{x+ct} \sin s\, ds = \cos(x+ct) - \cos(x-ct),$$

   and so we have

   $$u(x,t) = \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}(\cos(x+ct) - \cos(x-ct)).$$

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2. **(Strauss 2.1.7.)** We define an odd function to be any function f such that $f(-x) = -f(x)$ for all x. Prove that if that the initial conditions ϕ, ψ are odd functions, then so is the solution $u(x,t)$ for any fixed time t.

   Solution: There are two completely different solutions:

   - Again use d'Alembert. We know that the solution is

     $$u(x,t) = \frac{1}{2}(\phi(x+ct) + \psi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\, ds.$$

     Now, we compute

     $$u(-x,t) = \frac{1}{2}(\phi(-x+ct) + \psi(-x-ct)) + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\, ds$$

     Using the fact that ϕ is odd, we have

     $$\phi(-x-ct) = \phi(-(x+ct)) = -\phi(x+ct),$$
     $$\phi(-x+ct) = \phi(-(x-ct)) = -\phi(x-ct).$$

     Now consider the integral, and make a change of variables $r = -s$, giving

     $$\int_{-x-ct}^{-x+ct} \psi(s)\, ds = -\int_{x+ct}^{x-ct} \psi(-r)\, dr = -\int_{x-ct}^{x+ct} \psi(r)\, dr.$$

     (Note in the last equality there are actually three minus signs, since we use the fact that ψ is odd, *and* we flip the domain of integration.) Putting all of this together gives $u(-x,t) = -u(x,t)$ and thus u is odd.

   - We showed that reversing time gives a solution to the wave equation, and so does reversing space: if we choose $v(x,t) = -u(-x,t)$, then we see that

     $$v_{tt}(x,t) = -u_{tt}(-x,t), \quad v_{xx}(x,t) = -u_{xx}(-x,t),$$

     and thus if u satisfies

     $$u_{tt} = c^2 u_{xx}, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

then v satisfies

$$v_{tt} = c^2 v_{xx}, \quad v(x,0) = \phi(-x), \quad v_t(x,0) = \psi(-x).$$

This is true for any solution to the wave equation. If we now further assume that ϕ, ψ are odd, then these two PDE have the same initial data, and therefore by uniqueness, $v(x,t) = u(x,t)$ for all x, t, and thus $u(x,t) = -u(-x,t)$, and u is odd.

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3. We have defined a "well-posed" problem in class (also see book) typically for PDE, but we can consider if an ODE satisfies these three properties as well. Here you are given a sequence of ODEs and initial conditions; determine which of these problems are well-posed, and which are not[1]:

   (a) $\dfrac{dy}{dt} = 2y, \quad y(0) = 2,$

   (b) $\dfrac{dy}{dx} = \ln x, \quad y(0) = 0.$

   Solution:

   (a) This ODE satisfies the hypotheses of the E-U theorem for ODEs (in fact, the vector field is C^∞) and thus it is well-posed.

   (b) We cannot apply the theorem directly, but we can solve this ODE exactly. In fact, the general solution can be written as $y(x) = x \log x - x + C$ for some constant C. We see that there is exactly one solution with $y(0) = 0$ (in fact, choose $C = 0$). To show stability, we need to show that if we choose two different initial conditions, we can make the solutions close by choose the initial conditions close. So consider the two solutions

   $$y_1(x) = x \log x - x, \quad y_2(x) = x \log x - x + C.$$

   Clearly, $y_2(0) = C$, and by choosing $|y_2(0)| < \epsilon$, we can make $|y_1(x) - y_2(x)| < \epsilon$ for any x.

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4. **(Strauss 2.2.2.)** Let us consider a solution to the wave equation $u_{tt} = u_{xx}$ (we have assumed that $c^2 = 1$). Define the *energy density* $e(x,t) = \frac{1}{2}(u_t^2 + u_x^2)$ and the *momentum density* $p(x,t) = u_t u_x$. Show that

   (a) $\dfrac{\partial e}{\partial t} = \dfrac{\partial p}{\partial x}$ and $\dfrac{\partial p}{\partial t} = \dfrac{\partial e}{\partial x}$,

   (b) e and p both satisfy the wave equation themselves (although with different initial conditions).

   Solution:

   (a) We have

   $$\frac{\partial e}{\partial t} = u_t u_{tt} + u_x u_{xt}, \quad \frac{\partial p}{\partial x} = u_{tx} u_x + u_t u_{xx}.$$

   Using $u_{tt} = u_{xx}$ shows these are equal. (similar for the other)

   (b) We have

   $$\begin{aligned} e_x &= u_t u_{tx} + u_x u_{xx}, \\ e_{xx} &= u_{tx}^2 + u_t u_{txx} + u_{xx}^2 + u_x u_{xxx}, \\ e_t &= u_t u_{tt} + u_x u_{xt}, \\ e_{tt} &= u_{tt}^2 + u_t u_{ttt} + u_{tx}^2 + u_x u_{xtt} \end{aligned}$$

[1] Recall the Existence–Uniqueness Theorem which you saw in ODEs

Then

$$\begin{aligned} e_{xx} - e_{tt} &= u_t u_{txx} - u_t u_{ttt} + u_{xx}^2 - u_{tt}^2 + u_x u_{xxx} - u_x u_{ttx} \\ &= u_t(u_{xx} - u_{tt})_t + u_{xx}^2 - u_{tt}^2 + u_x(u_{xx} - u_{tt})_x = 0 + 0 + 0 = 0. \end{aligned}$$

(similar for p)

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5. **(Strauss 2.2.3.)** Show the following invariance properties for solutions of the wave equation. Assume that $u(x,t)$ satisfies the wave equation, then show that each of the transformed solutions *also* satisfy the wave equation:

   (a) **translation:** $u(x-\alpha, t)$ for any α,

   (b) **derivative:** $u_x(x,t)$,

   (c) **dilation:** $u(ax, at)$ for any a

   Solution:

   (a) Define $v(x,t) = u(x-\alpha, t)$. We see, using the chain rule, that

   $$\frac{\partial v}{\partial x}(x,t) = \frac{\partial u}{\partial x}(x-\alpha,t) \cdot \frac{d}{dx}(x-\alpha) = \frac{\partial u}{\partial x}(x-\alpha,t).$$

   Similarly,

   $$\frac{\partial^2 v}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x-\alpha,t).$$

   We also work out that

   $$\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x-\alpha,t).$$

   Therefore

   $$v_{tt}(x,t) - c^2 v_{xx}(x,t) = u_{tt}(x-\alpha,t) - c^2 u_{xx}(x-\alpha,t) = 0.$$

   (b) Define $v(x,t) = u_x(x,t)$. Then

   $$v_{tt} = u_{xtt}, \quad v_{xx} = u_{xxx},$$

   so

   $$v_{tt} - c^2 v_{xx} = u_{xtt} - c^2 u_{xxx} = u_{ttx} - c^2 u_{xxx} = (u_{tt} - c^2 u_{xx})_x = \frac{\partial 0}{\partial x} = 0.$$

   (c) Define $v(x,t) = u(ax, at)$. Then

   $$\begin{aligned} \frac{\partial v}{\partial x}(x,t) &= \frac{\partial u}{\partial x}(ax,at) \cdot a, \\ \frac{\partial^2 v}{\partial x^2}(x,t) &= \frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}(ax,at)) = a\frac{\partial^2 u}{\partial x^2}(ax,at) \cdot a = a^2 \frac{\partial^2 u}{\partial x^2}(ax,at), \\ \frac{\partial v}{\partial t}(x,t) &= \frac{\partial u}{\partial t}(ax,at) \cdot a, \\ \frac{\partial^2 v}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}(a\frac{\partial u}{\partial t}(ax,at)) = a\frac{\partial^2 u}{\partial t^2}(ax,at) \cdot a = a^2 \frac{\partial^2 u}{\partial t^2}(ax,at). \end{aligned}$$

   Thus we have

   $$v_{tt}(x,t) - c^2 v_{xx}(x,t) = a^2 u_{tt}(ax,at) - a^2 c^2 u_{xx}(ax,at) = a^2(u_{tt}(ax,at) - c^2 u_{xx}(ax,at)) = a^2 \cdot 0 = 0.$$

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6. **(Strauss 2.3.3.)** Consider a solution to the diffusion equation $u_t = u_{xx}$ for $x \in [0, L]$ and $t > 0$. Define

$$M(T) = \text{maximum of } u(x,t) \text{ on the rectangle } [0,L] \times [0,T],$$
$$m(T) = \text{minimum of } u(x,t) \text{ on the rectangle } [0,L] \times [0,T].$$

Does $M(T)$ increase or decrease as a function of T? Does $m(T)$ increase or decrease as a function of T? Explain why.

Solution: It turns out the answer to this is somewhat complicated as it could depend on the boundary and initial conditions. Let us first fix the boundary conditions as

$$u(0,t) = 0, \quad u(L,t) = 0,$$

for all t and assume that the initial condition $\phi(x)$ is positive for some $x \in [0, L]$ so that $M(0) > 0$. The claim here then is that $M(T), m(T)$ are functions which are constant in T.

First, notice that by definition, if $T' > T$, then $M(T') \geq M(T)$ (since we're taking the maximum over a larger set).

We now want to prove that $M(T') \leq M(T)$, and then we are done. So we prove by contradiction: assume that $M(T') > M(T)$. If this is so, then clearly the maximum inside the rectangle $[0, L] \times [0, T']$ must occur for $t \in (T, T']$. Since $M(T') > M(T) \geq M(0) > 0$, this means that this maximum may not occur on the left- or right-hand edges, but must occur either in the interior of the rectangle, or on the top edge. But this directly contradicts the Maximum Principle. Since assuming $M(T') > M(T)$ leads to a contradiction, it must be true that $M(T') \leq M(T)$.

The argument for $m(T)$ is similar: reverse every inequality above and use the Minimum Principle.

Of course, with different boundary conditions things could be more complicated. If we now assume that $u(0,t) = f(t)$, and this function is increasing (rapidly enough), then it's possible that $M(T)$ increases, since $M(T) \geq \max_{t\in[0,T]} f(t)$ at least. In this case, the final statement would be that $M(T)$ can increase, but no faster than f. Other permutations are left to the reader...