Model problem

- Can we find an integer solution to the equation \(4x + 6y = 1\)?
  - After some reflection, the answer is no: because the left-hand side is even, you can never get 1 on the right-hand side.

- Ok what about \(4x + 6y = 2\)?
  - Yes we can! For example, \(x = 2, y = -1\) works.
  - Ok, is this the only solution? (Probably not.)
  - Can we find all the solutions???
Definition

- Let $m, n \in \mathbb{Z}$ and not both zero. Then the **greatest common divisor** of $m, n$, denote $\gcd(m, n)$, is the largest positive number that divides both of them.
- If $\gcd(m, n) = 1$ we say that $m$ and $n$ are relatively prime.

Example

Let $m = 42$ and $n = 63$.

- **Divisors of** $42$: $1, 2, 3, 6, 7, 14, 21, 42$
- **Divisors of** $63$: $1, 3, 7, 9, 21, 63$

Then $\gcd(42, 63) = 21$. 
We want to find \( \gcd(m, n) \). Say \( n < m \). Then we use the Division Algorithm to obtain

\[ m = q_1 n + r_1. \]

If \( r_1 = 0 \), we are done, because \( n \mid m \) and the answer is \( n \).

If not, we solve

\[ n = q_2 r_1 + r_2. \]

If \( r_2 = 0 \), the answer is \( r_1 \). If not, keep going:

\[ r_1 = q_3 r_2 + r_3 \]

etc.

When this algorithm terminates, the last non-zero remainder is the \( \gcd \)!!!
Finding gcd(2163, 824)

- We compute:

\[
2163 = 2 \times 824 + 515, \\
824 = 1 \times 515 + 309, \\
515 = 1 \times 309 + 206, \\
309 = 1 \times 206 + 103, \\
206 = 2 \times 103 + 0,
\]

so the gcd is 103.

- In fact, 2163 = 21 \times 103 and 824 = 8 \times 103.

We can also write

\[
103 = 1 \times 309 - 1 \times 206 \\
= 1 \times 309 - 1(515 - 309) = 2 \times 309 - 1 \times 515 \\
= 2(824 - 515) - 1 \times 515 = 2 \times 824 - 3 \times 515 \\
= 2 \times 824 - 3(2163 - 2 \times 824) = 8 \times 824 - 3 \times 2163
\]
Bézout’s Identity

Theorem (Bézout’s Identity)

For any \( m, n \) not both zero, there exists \((x, y)\), an integer solution of the equation

\[ mx + ny = \gcd(m, n). \]

The basic idea is to backpropagate the Euclidean algorithm.
Choose $m = 2163$ and $n = 824$ and we ask if we can solve:

$$2163x + 824y = 103.$$  \hspace{1cm} (1)

We know we can, because we did it two slides ago! In fact we have $x = -3, y = 8$ as a particular solution.

Can we find all the solutions of (1)?

Divide (1) by 103 and we get (the same equation!)

$$21x + 8y = 1.$$  \hspace{1cm} (2)

Now ask: can we solve this equation when the right-hand side is zero? Yes!

$x = 8t, y = -21t$ solves $21x + 8y = 0$ for any integer $t$.

Therefore we pick

$$x = -3 + 8t, y = 8 - 21t.$$  

Check:

$$21(-3 + 8t) + 8(8 - 21t) = -63 + 168t + 64 - 168t = 1.$$  \hspace{1cm} (3)