Main goal today: Cover propositions, and how to “combine” them...
Propositions - definition

Definition

A proposition is a sentence that is (objectively) true or false.

That means...

1. It has to be one or the other! Not “both”, not “neither”
2. It can’t be subjective, or an opinion...
3. However!! it is ok if we don’t know the answer
### Propositions - Examples

#### Examples

1. “3 + 4 = 7” is a (true) proposition;
2. “8 − 9 = 405” is a (false) proposition;
3. “There are extraterrestrial aliens on Earth right now” is a proposition.

#### Non-examples

1. “Compute 4 + 3” is not a proposition;
2. “The best number is \( \pi \)” is not a proposition;
3. “The best restaurant on Green St. is Mid-summer Lounge” is not a proposition.
Let $P$ and $Q$ be propositions, then we define

1. “$P$ AND $Q$”, $P \land Q$, is true only if both $P$ and $Q$ are true;
2. “$P$ OR $Q$”, $P \lor Q$, is true if either or both of $P$ and $Q$ are true;
3. “NOT $P$”, $\neg P$, that is true if $P$ is false and false if $P$ is true;
4. “$P$ XOR $Q$”, $P \oplus Q$, is true if one but not both of $P$ and $Q$ are true.
Describing this with Truth Tables

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### XOR

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Tautologies and contradictions

**Definition**

A **tautology** is a proposition that is always true.

A **contradiction** is a proposition that is always false.

- \( P \lor \neg P \) is a tautology;
- \( Q \land \neg Q \) is a contradiction;
- \( ((P \lor Q) \land \neg P) \land \neg Q \) is a tautology (see video short working this out!).
Conditionals

**Definition**

We define $P \implies Q$ to be the proposition that is true unless $P$ is true and $Q$ is false, namely:

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We typically read this as “if $P$, then $Q$”, but can also read it as

- “$Q$ if $P$”;
- “$P$ only if $Q$”;
- “$P$ is sufficient for $Q$”;
- “$Q$ is necessary for $P$”.

Note: $(P \implies Q)$ is **logically** equivalent to $(\neg P \lor Q)$ (Check!!).
Conditional, examples:

Let \( P = \text{“Ben is from Chicago”} \) and \( Q = \text{“Ben is from Illinois”} \).

The various ways of writing \( P \implies Q \):  
- If Ben is from Chicago, then Ben is from Illinois;  
- Ben is from Illinois if Ben is from Chicago;  
- Ben is from Chicago only if Ben is from Illinois;  
- Ben being from Chicago is sufficient for Ben being from Illinois;  
- Ben being from Illinois is necessary for Ben being from Chicago.

These are not always easy to parse — so it is useful to convert to the precise \( P \implies Q \).
Ok, so there is one weird thing about the conditional, and it is this: $F \implies T$ is actually true!

Here’s why it is weird. In this system:

1. “24 is even $\implies$ the moon is made of cheese” is false;
2. “24 is odd $\implies$ the moon is made of cheese” is true!

However, we do it this way because of reasons. (which I hope will become clear throughout the course)
Biconditionals

**Definition**

We define $P \iff Q$ to be the proposition that is true when $P$ and $Q$ are the same, namely:

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We typically read this as “$P$ if and only if $Q$”, but can also read it as

- “$P$ iff $Q$”;
- “$P$ is necessary and sufficient for $Q$”;
- “$P$ and $Q$ are (logically) equivalent”.

Note: $(P \iff Q)$ is **logically** equivalent to:

1. $\neg(P \oplus Q)$;
2. $(P \implies Q) \land (Q \implies P)$. 
For any implication \( P \Rightarrow Q \), there are four related statements:

- **positive:** \( P \Rightarrow Q \);
- **converse:** \( Q \Rightarrow P \);
- **inverse:** \( \neg P \Rightarrow \neg Q \);
- **contrapositive:** \( \neg Q \Rightarrow \neg P \).

The positive and the contrapositive are **equivalent**.

**Note!!!!** The positive and the converse and **not equivalent!!!!!!!1!**
De Morgan’s Laws

- \( \neg(P \land Q) \iff \neg P \lor \neg Q \);
- \( \neg(P \lor Q) \iff \neg P \land \neg Q \).

We also have

- **commutativity:**
  - \( P \lor Q \iff Q \lor P \);
  - \( P \land Q \iff Q \land P \);

- **associativity:**
  - \( (P \lor Q) \lor R \iff P \lor (Q \lor R) \);
  - \( (P \land Q) \land R \iff P \land (Q \land R) \);

- **distributivity:**
  - \( (P \land Q) \lor R \iff (P \lor R) \land (Q \lor R) \);
  - \( (P \lor Q) \land R \iff (P \land R) \lor (Q \land R) \).