Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 9 — due November 11

1. Determine the fundamental period of the following functions:
   (a) \( \cos(2t) \)
   (b) \( \sin(2\pi t) \)
   (c) \( \sin^2(t) \)
   (d) \( \cos(t) + \sin(t) \)

Solution: We know from trig that the fundamental period of \( \cos(t) \) or \( \sin(t) \) is \( 2\pi \). Clearly, then, if we consider \( \cos(\alpha t) \) or \( \sin(\alpha t) \) for any real \( \alpha \), then this should have fundamental period \( 2\pi/\alpha \). To see this, notice that \( \cos(\alpha t) \) has a period of \( 2\pi/\alpha \):

\[
\cos(\alpha(t + 2\pi/\alpha)) = \cos(\alpha t + 2\pi) = \cos(\alpha t),
\]
and, moreover, if \( p \) is a period for \( \cos(\alpha t) \), then we have

\[
\cos(\alpha(t + p)) = \cos(\alpha t + \alpha p) = \cos(\alpha t)
\]
for all \( t \), and therefore \( \alpha p = 2k\pi \) for some integer \( k \), and thus \( p = 2k\pi/\alpha \) for some integer \( k \). The smallest of these is choosing \( k = 1 \), or \( 2\pi/\alpha \).

Once we know all this, solving (a,b) is straightforward, and we obtain \( \pi \) and 1, respectively.

For (c), we see that clearly any period of \( \sin(t) \) is also a period of \( \sin^2(t) \), i.e. if we have a \( p \) with \( \sin(t + p) = \sin(t) \) for all \( t \), then clearly, also \( \sin^2(t + p) = \sin^2(t) \) for all \( t \). However, it is possible that \( \sin^2(t) \) has a smaller period, because if we require that

\[
\sin^2(t + p) = \sin^2(t)
\]
then this means that

\[
\sin(t + p) = \pm \sin(t)
\]
If it is possible to solve this equation with the minus sign with a \( p \) smaller than \( 2\pi \), then we have a smaller fundamental period. But note that

\[
\sin(t + \pi) = -\sin(t)
\]
for all \( t \), and thus

\[
\sin^2(t + \pi) = \sin^2(t)
\]
and we have a smaller fundamental period.

Part (d) is a bit trickier. We need a \( p \) which solves

\[
\cos(t + p) + \sin(t + p) = \cos(t) + \sin(t)
\]
for all \( t \). Now, of course, any multiple of \( 2\pi \) will work, but can we do it with a smaller \( p \)? We use the trig identities

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B,
\]
\[
\sin(A + B) = \cos A \sin B + \sin A \cos B.
\]
Thus we have
\[
\cos(t + p) + \sin(t + p) = \cos t \cos p - \sin t \sin p + \cos t \sin p + \sin t \cos p
\]
\[
= (\cos p + \sin p) \cos t + (\cos p - \sin p) \sin t.
\]
Thus we need
\[
\cos p + \sin p = 1,
\]
\[
\cos p - \sin p = 0,
\]
or \(\cos p = 1, \sin p = 0\). This is solved by exactly \(p = 2k\pi\), and thus \(2\pi\) is the fundamental period.

2. Is the function \(f(t) = \cos(t) + \cos(4t)\) periodic? If yes, demonstrate this by finding a period of the function. Same questions for \(g(t) = \cos(t) + \cos(\pi t)\).

**Solution:** Yes it is, and the way to see that is to note that the periods of \(\cos(t)\) are \(2k\pi\), and the periods of \(\cos(4t)\) are \(l\pi/2\), where \(k, l\) are integers. These sets of numbers share a common element, namely \(2\pi\) (choose \(k = 1, l = 4\)) and thus \(f(t)\) is periodic with period \(2\pi\).

On the other hand, this will not work for \(g\); notice that the periods of \(\cos(t)\) are \(2k\pi\), but the periods of \(\cos(\pi t)\) are \(2l\), and
\[
2k\pi = 2l
\]
only if \(\pi = l/k\), which would mean \(\pi\) is rational, which it is not.

3. Let \(f(t)\) be a \(2\pi\)-periodic function defined by
\[
f(t) = \begin{cases} 
3, & -\pi < t < 0, \\
-4, & 0 < t < \pi, \\
132, & t = 0, \pi.
\end{cases}
\]
Compute its Fourier series.

**Solution:** We know that the 132 in the formula will not matter at all, so we can ignore it. Here we have \(L = \pi\).

We use the formula for
\[
A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt
\]
\[
= \frac{1}{\pi} \left( \int_{-\pi}^{0} 3 \cos(nt) \, dt + \int_{0}^{\pi} -4 \cos(nt) \, dt \right),
\]
\[
= \frac{1}{\pi} \left( \frac{3}{n} \sin(nt) \bigg|_{t=-\pi}^{t=0} + \frac{-4}{n} \sin(nt) \bigg|_{t=0}^{t=\pi} \right)
\]
\[
= \frac{1}{\pi} (0 - 0 + 0 - 0) = 0,
\]
but this formula only works for \(n > 0\) since we divided by \(n\). For
\[
A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} (-\pi) = -1.
\]
For $B_n$, we compute

\[ B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt \]
\[ = \frac{1}{\pi} \left( \int_{-\pi}^{0} 3 \sin(nt) \, dt + \int_{0}^{\pi} -4 \sin(nt) \, dt \right) , \]
\[ = \frac{1}{\pi} \left( \frac{-3}{n} \cos(nt) \bigg|_{t=0}^{t=\pi} + \frac{4}{n} \cos(nt) \bigg|_{t=\pi}^{t=0} \right) \]
\[ = \frac{1}{n\pi} (-3(1 - (-1)^n) + 4((-1)^n - 1)), \]

which is 0 when $n$ is even, but $-14/n$ when $n$ is odd. Therefore the Fourier series of $f$ is

\[ -\frac{1}{2} + \sum_{n \text{ odd}} -\frac{14}{n} \sin(nt). \]

4. Let $f(t)$ be a $2\pi$-periodic function defined by $f(t) = |t|$ for $t \in [-\pi, \pi]$ and extended periodically elsewhere. Compute its Fourier series.

**Solution:** We first note that $f$ is even and therefore all of the $B_n$ are zero. Moreover, we can use the Fourier cosine series coefficient formula for $A_n$ and save a bit of writing, so we have

\[ A_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos(nt) \, dt \]
\[ = \frac{2}{\pi} \int_{0}^{\pi} t \cos(nt) \, dt, \]
\[ = \frac{2}{\pi} \left( \frac{t \sin(nt)}{n} \bigg|_{t=0}^{t=\pi} - \int_{0}^{\pi} \frac{1}{n} \sin(nt) \, dt \right) \]
\[ = \frac{2}{\pi} \left( \frac{t \sin(nt)}{n} \bigg|_{t=0}^{t=\pi} - \frac{\cos(nt)}{n^2} \bigg|_{t=0}^{t=\pi} \right) = \frac{2}{\pi} \begin{cases} \frac{\pi}{2}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \]

where, again, this does not work if $n = 0$. For $A_0$ we compute

\[ A_0 = \frac{2}{\pi} \int_{0}^{\pi} t \, dt = \frac{\pi}{2}. \]

Thus the Fourier series is

\[ \frac{\pi}{2} + \sum_{n \text{ odd}} -\frac{4}{n^2} \cos(nt). \]

5. Define $f$ to be the function with period 3 defined as

\[ f(t) = t^2, \quad -3/2 < t < 3/2. \]

Compute its Fourier series.
Solution: Again note that \( t^2 \) is even so we need not compute \( B_n \). We have

\[
A_n = \frac{2}{3} \int_{-3/2}^{3/2} t^2 \cos(2\pi nt/3) \, dt.
\]

To simplify notation, we first compute

\[
\int t^2 \cos(n\pi t/L) \, dt,
\]

and after two integrations by parts, we obtain

\[
\int t^2 \cos(n\pi t/L) \, dt = -\frac{t^2L}{n\pi} \sin(n\pi t/L) + \frac{2tL^2}{n^2\pi^2} \cos(n\pi t/L) - \frac{2L^3}{n^3\pi^3} \sin(n\pi t/L).
\]

Computing the definite integral (evaluating all of these terms at \( L \) and \(-L\)) gives

\[
0 - 0 + \frac{2L^3}{n^2\pi^2} \cos(n\pi) + \frac{2L^3}{n^2\pi^2} \cos(-n\pi) = (-1)^n \frac{4L^3}{n^2\pi^2}.
\]

Of course, we also have to do the separate calculation

\[
A_0 = \frac{1}{L} \int_{-L}^{L} t^2 \, dt = \frac{1}{L} \frac{t^3}{3} \bigg|_{t=-L}^{t=L} = \frac{2}{3} L^2.
\]

Plugging in \( L = 3/2 \) gives

\[
A_0 = \frac{3}{2},
\]

\[
A_n = (-1)^n \frac{27}{2n^2\pi^2}.
\]

So we have a Fourier series of

\[
\frac{3}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{27}{2n^2\pi^2} \cos(2n\pi t/3).
\]

6. Prove that

\[
\int_{-\pi}^{\pi} \cos(nt) \sin(mt) \, dt = 0
\]

for any integers \( n, m \). Hint: Think about even and odd functions.

Solution: \( \cos \) is even and \( \sin \) is odd, therefore their product is odd, therefore the integral over any symmetric interval is zero.

7. Prove that

\[
\int_{-\pi}^{\pi} \cos(nt) \cos(mt) \, dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}
\]

Hint: For \( m \neq n \), use the trig identity

\[
\cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B)).
\]

Why does this calculation fail when \( m = n \)?
Solution: We compute

\[
\int_{-\pi}^{\pi} \cos(nt) \cos(mt) = \frac{1}{2} \int_{-\pi}^{\pi} \cos((m + n)t) + \cos((m - n)t) \, dt
\]

\[
= \frac{1}{2} \left[ \frac{\sin((m + n)t)}{m + n} + \frac{\sin((m - n)t)}{m - n} \right]_{t=-\pi}^{t=\pi}
\]

\[
= \frac{1}{2} \left( 0 - 0 + 0 - 0 \right) = 0,
\]

but notice that we divided by \(m - n\), so this formula is invalid for \(m = n\).