1. Solve the initial value problem

\[ y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 2, \]

and now solve

\[ y'' - y = 0, \quad y(0) = 0, \quad y'(0) = -1. \]

**Solution:** We make the exponential Ansatz \( y(x) = e^{rx} \), which leads to the equation

\[ r^2 - 1 = 0, \]

which has roots \( r = \pm 1 \). Therefore, two solutions to this system are \( y_1(x) = e^x \) and \( y_2(x) = e^{-x} \), and so the general solution to this system is

\[ y(x) = C_1 e^x + C_2 e^{-x}. \]

We also have

\[ y'(x) = C_1 e^x - C_2 e^{-x}, \]

so

\[ y(0) = C_1 + C_2, \quad y'(0) = C_1 - C_2. \]

To solve the first problem we obtain

\[ C_1 + C_2 = 1, \]
\[ C_1 - C_2 = 2. \]

which we solve as \( C_1 = 3/2, C_2 = -1/2 \), so the solution is

\[ y(x) = \frac{3}{2} e^x - \frac{1}{2} e^{-x}. \]

For the second problem, we obtain

\[ C_1 + C_2 = 0, \]
\[ C_1 - C_2 = -1. \]

which we solve as \( C_1 = -1/2, C_2 = 1/2 \), so the solution is

\[ y(x) = -\frac{1}{2} e^x + \frac{1}{2} e^{-x}. \]

2. Solve the initial value problem

\[ y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 2, \]

and now solve

\[ y'' + y = 0, \quad y(0) = -1, \quad y'(0) = 0. \]
Solution: Making the Ansatz $y(x) = e^{rx}$ and plugging in gives
$$r^2 + 1 = 0,$$
which has roots $\pm i$. Thus our two solutions are $\cos(x)$ and $\sin(x)$, and the general solution is
$$y(x) = C_1 \cos(x) + C_2 \sin(x).$$
We also have
$$y'(x) = -C_1 \sin(x) + C_2 \cos(x),$$
so
$$y(0) = C_1, \quad y'(0) = C_2.$$
Thus the first problem has $C_1 = 1, C_2 = 2$, or
$$y(x) = \cos(x) + 2\sin(x).$$
The second problem has $C_1 = -1, C_2 = 0$, or
$$y(x) = -\cos(x).$$

3. In each of the following problems, you should give the general solution of the differential equation (i.e. do steps 1 & 2 as described in class)

(a) $y'' - 2y' + y = 0,$
(b) $y'' - 3y' + y = 0,$
(c) $y'' - y' + y = 0,$
(d) $y' - 2y = 0,$
(e) $y'' - 2y' = 0,$
(f) $y''' - 2y'' = 0.$

Solution: In each case, we compute the characteristic equation and then find the roots which gives us the solutions.

(a) $r^2 - 2r + 1$ has repeated root $r = 1, 1$, so our two solutions are $e^x, xe^x$, and the general solution is
$$y(x) = C_1 e^x + C_2 xe^x.$$

(b) We get $r^2 - 3r + 1 = 0$. Using the quadratic formula gives
$$r = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2},$$
so our general solution is
$$y(x) = C_1 e^{(3+\sqrt{5})x/2} + C_2 e^{(3-\sqrt{5})x/2}.$$

(c) We get $r^2 - r + 1 = 0$, and using the quadratic formula gives
$$\frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2},$$
Thus our general solution is
$$y(x) = C_1 e^{x/2} \cos(3x/2) + C_2 e^{x/2} \sin(3x/2).$$
(d) We get \( r - 2 = 0 \), which has one root, \( r = 2 \), so our general solution is
\[
y(x) = Ce^{2x}.
\]

(e) We get \( r^2 - 2r = 0 \), which has roots \( r = 0, 2 \), so we get
\[
y(x) = C_1 e^{0x} + C_2 e^{2x} = C_1 + C_2 e^{2x}.
\]

(f) We get \( r^3 - 2r^2 = 0 \), which has roots \( r = 0, 0, 2 \), so our three solutions should be
\[
y_1(x) = e^{2x}, \quad y_2(x) = e^{0x}, \quad y_3(x) = xe^{0x},
\]
so we obtain
\[
y(x) = C_1 e^{2x} + C_2 + C_3 x.
\]